A sublinear time quantum algorithm for approximating partition functions

arXiv:2207.08643

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August 24th, 2023

INSTITUT DE RECHERCHE EN INFORMATIQUE FONDAMENTALE

SIMONS INSTITUTE for the Theory of Computing

Volume estimation

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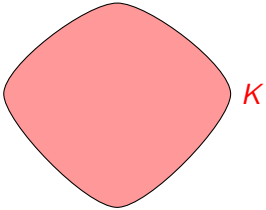
- *Convex body:* $K \subseteq \mathbb{R}^d$



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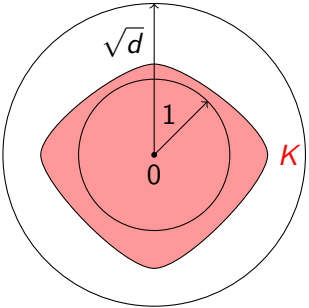

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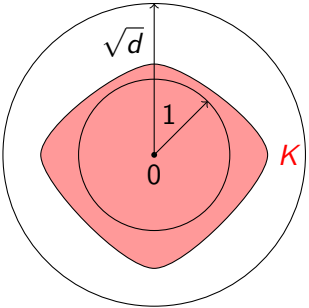

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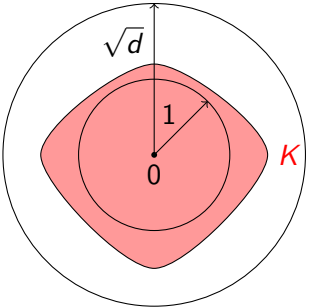

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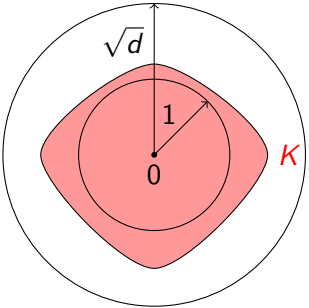

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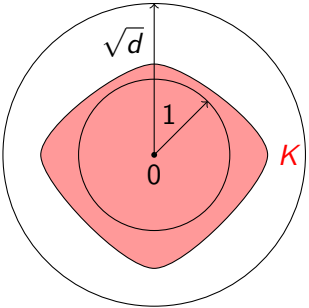

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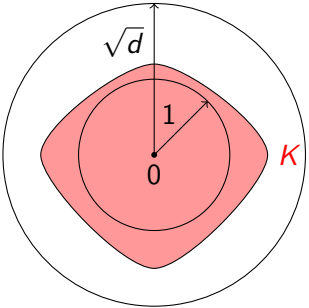
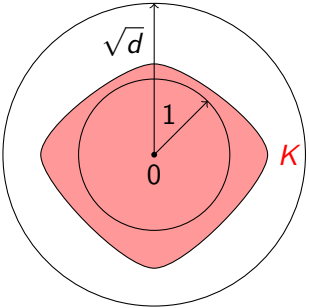
Classically		**Quantumly**	
[DF91]	$\widetilde{O}(\frac{d^8}{\varepsilon^2})$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	[CCH+19]	$\widetilde{O}(\frac{d^{2.5}}{\varepsilon})$
[CV15]	$\widetilde{O}(\frac{d^3}{\varepsilon^2})$	*[CH22]*	$\widetilde{O}(\frac{d^{2.25}}{\varepsilon})$



Partition functions



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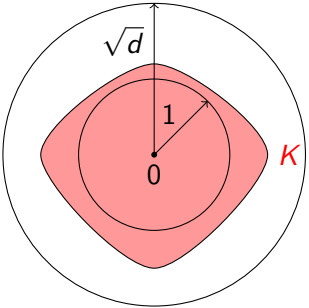

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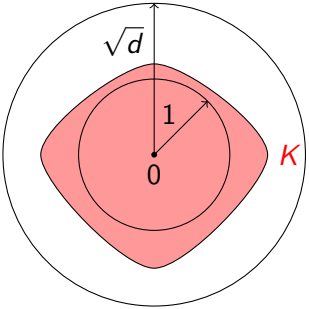

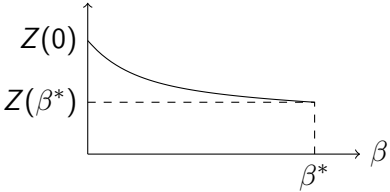

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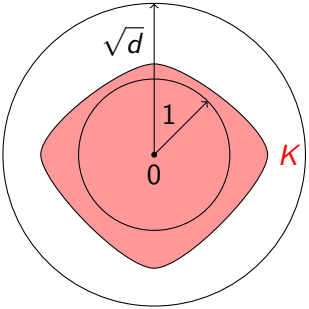

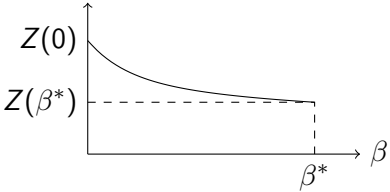

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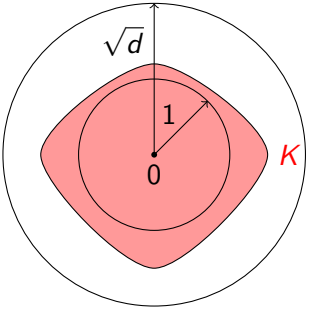

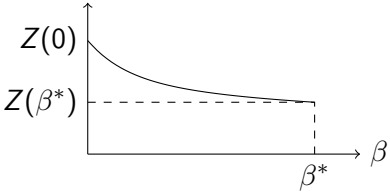

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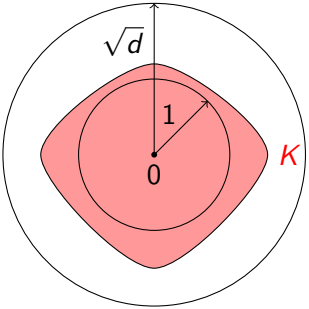

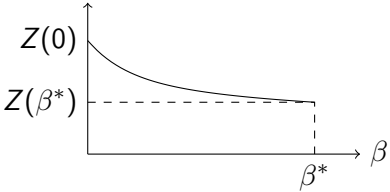

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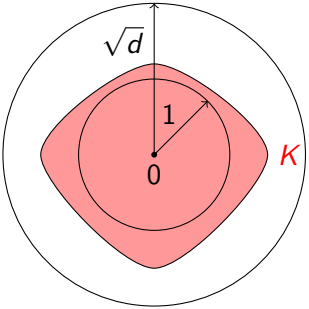
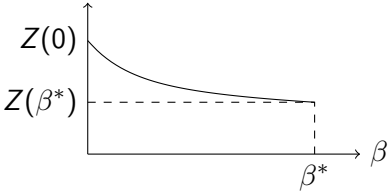
  - Many graph applications:
    - Counting independent sets.
    - Counting k-colorings.
    - Counting matchings.





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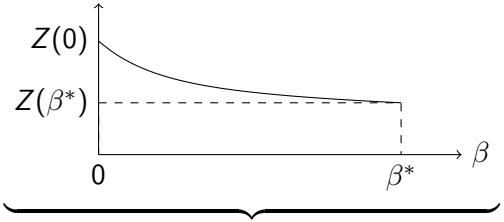

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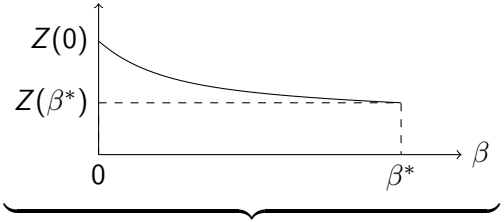

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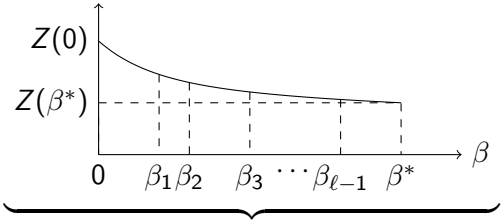

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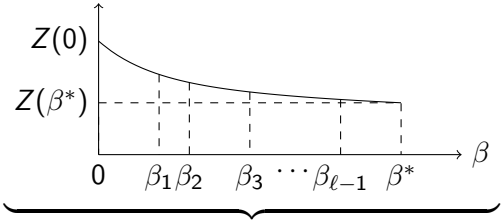

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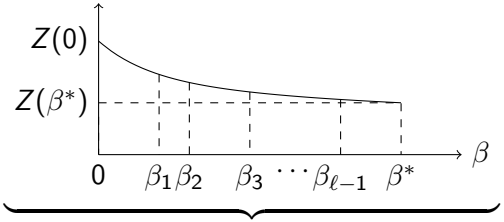

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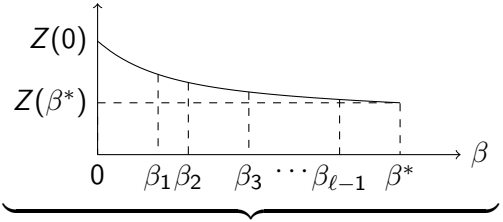

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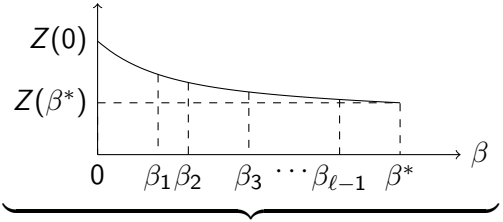

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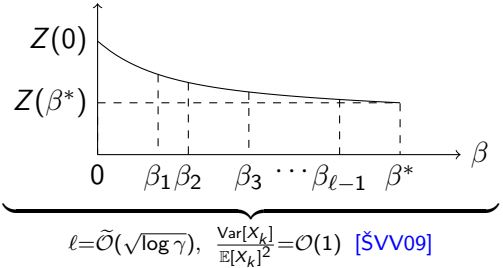

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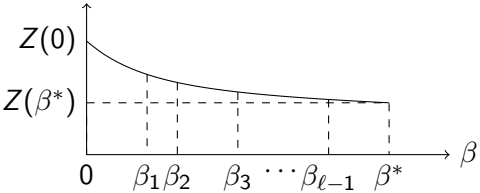
Cost

Classical	$\mathcal{O}\left(\ell \cdot \frac{\ell}{\varepsilon^2} \cdot \mathrm{MT}\right)$
Quantum, [HW20]	$\widetilde{\mathcal{O}}\left(\ell \cdot \sqrt{\frac{\ell^2}{\varepsilon^2} \cdot \mathrm{MT}}\right)$
Quantum, *[CH22]*	$\widetilde{\mathcal{O}}\left(\ell \cdot \sqrt{\frac{\ell}{\varepsilon^2} \cdot \mathrm{MT}}\right)$



$\ell = \widetilde{\mathcal{O}}(\sqrt{\log\gamma})$, $\frac{\mathrm{Var}[X_k]}{\mathbb{E}[X_k]^2} = \mathcal{O}(1)$ [ŠVV09]

Implications

	Classical	Quantum, [HW20]	Quantum, *[CH22]*
	$\mathcal{O}(\ell \cdot \frac{\ell}{\varepsilon^2} \cdot \mathrm{MT})$	$\widetilde{\mathcal{O}}\left(\ell \cdot \sqrt{\frac{\ell^2}{\varepsilon^2} \cdot \mathrm{MT}}\right)$	$\widetilde{\mathcal{O}}\left(\ell \cdot \sqrt{\frac{\ell}{\varepsilon^2} \cdot \mathrm{MT}}\right)$
General ($\ell = \widetilde{\mathcal{O}}(\sqrt{\log \gamma})$))	$\widetilde{\mathcal{O}}\left(\frac{\log \gamma}{\varepsilon^2} \cdot \mathrm{MT}\right)$	$\widetilde{\mathcal{O}}\left(\frac{\log \gamma}{\varepsilon} \cdot \sqrt{\mathrm{MT}}\right)$	$\widetilde{\mathcal{O}}\left(\frac{\log^{3/4} \gamma}{\varepsilon} \cdot \sqrt{\mathrm{MT}}\right)$

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Counting independent sets	$\widetilde{\mathcal{O}}(\frac{\lvert V\rvert^2}{\varepsilon^2})$	$\widetilde{\mathcal{O}}(\frac{\lvert V\rvert^{1.5}}{\varepsilon})$	$\widetilde{\mathcal{O}}(\frac{\lvert V\rvert^{1.25}}{\varepsilon})$
Counting graph colorings	$\widetilde{\mathcal{O}}(\frac{\lvert V\rvert^2}{\varepsilon^2})$	$\widetilde{\mathcal{O}}(\frac{\lvert V\rvert^{1.5}}{\varepsilon})$	$\widetilde{\mathcal{O}}(\frac{\lvert V\rvert^{1.25}}{\varepsilon})$
Counting graph matchings	$\widetilde{\mathcal{O}}(\frac{\lvert V\rvert\lvert E\rvert}{\varepsilon^2})$	$\widetilde{\mathcal{O}}(\frac{\lvert V\rvert\lvert E\rvert^{0.5}}{\varepsilon})$	$\widetilde{\mathcal{O}}(\frac{\lvert V\rvert^{0.75}\lvert E\rvert^{0.5}}{\varepsilon})$
Volume estimation well-rounded convex body	$\widetilde{\mathcal{O}}(\frac{d^3}{\varepsilon^2})$	$\widetilde{\mathcal{O}}(\frac{d^{2.5}}{\varepsilon})$	$\widetilde{\mathcal{O}}(\frac{d^{2.25}}{\varepsilon})$
$\vdots$			

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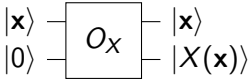

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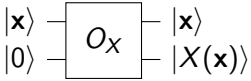

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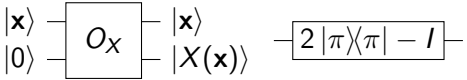

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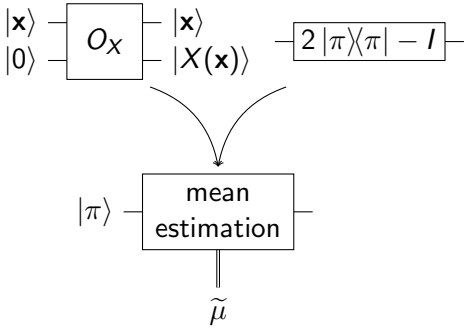

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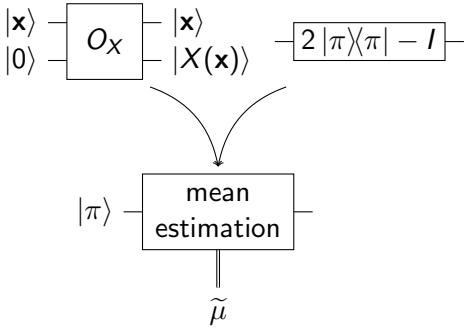

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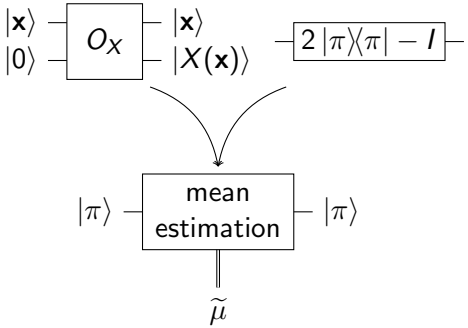

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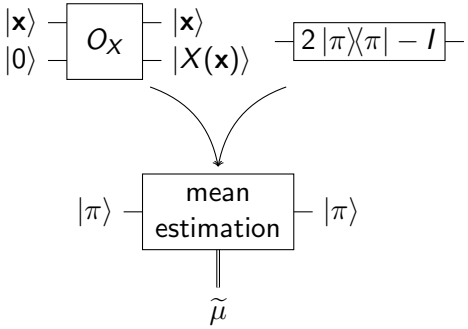
Cost: $\widetilde{O}(\frac{1}{\varepsilon})$ reflections around $|\pi\rangle$.



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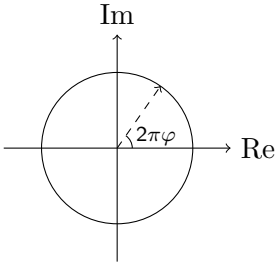

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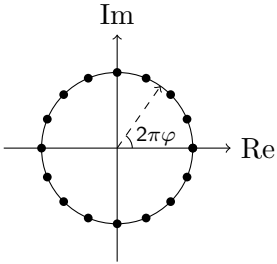

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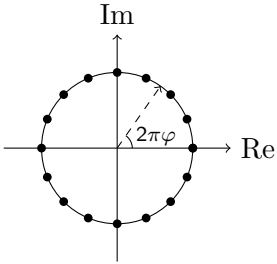

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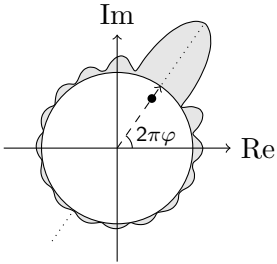

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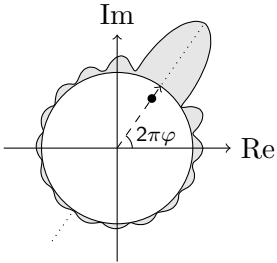

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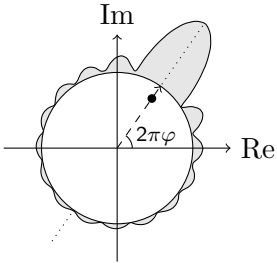
- Lift to mean estimation routines [Mon15].



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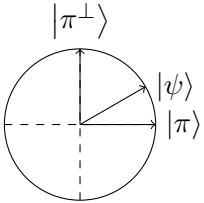

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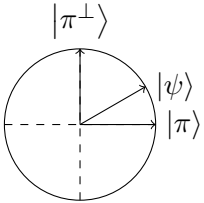

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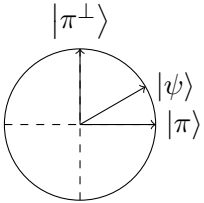

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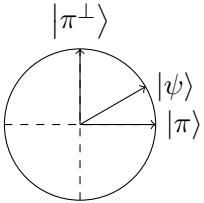

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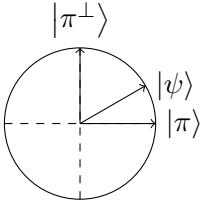

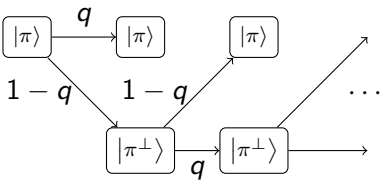

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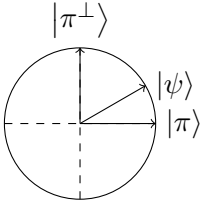

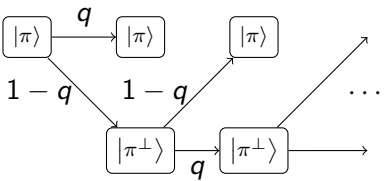

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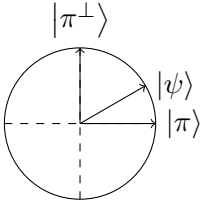

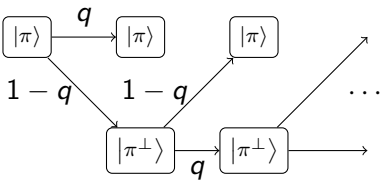

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Quantum, prev.	$\widetilde{\mathcal{O}}\left(\ell \cdot \sqrt{\frac{\ell^2}{\varepsilon^2} \cdot \mathrm{MT}}\right)$
Quantum, new	$\widetilde{\mathcal{O}}\left(\ell \cdot \sqrt{\frac{\ell}{\varepsilon^2} \cdot \mathrm{MT}}\right)$

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	Cost
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Quantum, prev.	$\widetilde{\mathcal{O}}\left(\ell \cdot \sqrt{\frac{\ell^2}{\varepsilon^2} \cdot \mathrm{MT}}\right)$
Quantum, new	$\widetilde{\mathcal{O}}\left(\ell \cdot \sqrt{\frac{\ell}{\varepsilon^2} \cdot \mathrm{MT}}\right)$

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- Algorithm outline.
- Results.
- New component: mean estimation.
  - Accuracy.
  - Unbiasedness.
  - Non-destructiveness.

Cost

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Open questions:

- Improve dependence on ℓ?
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Thanks for your attention!

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