Quantum algorithms through composition of graphs

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SIMONS INSTITUTE for the Theory of Computing

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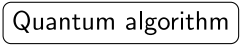

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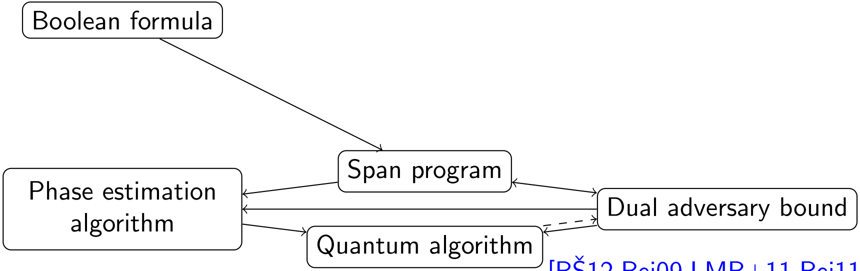

[RŠ12,Rei09,LMR+11,Rei11]

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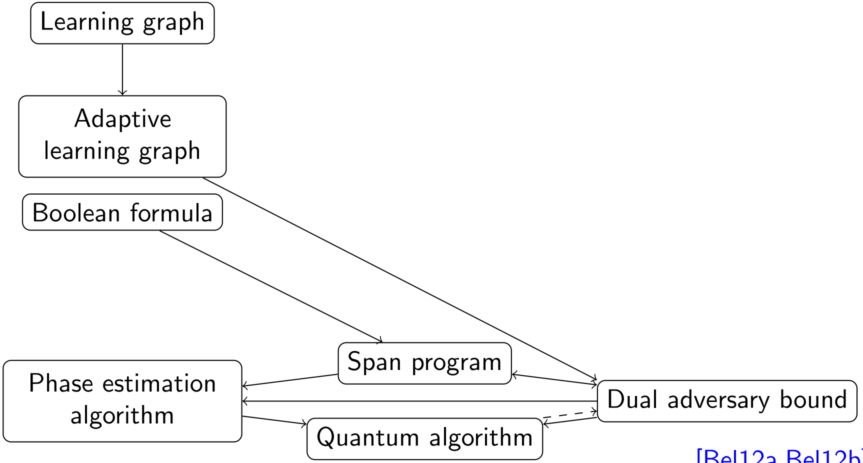

[Bel12a,Bel12b]

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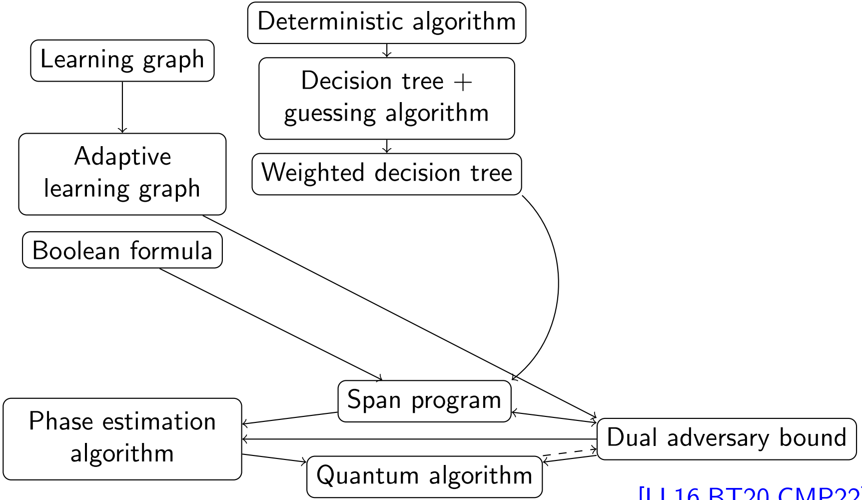

[LL16,BT20,CMP22]

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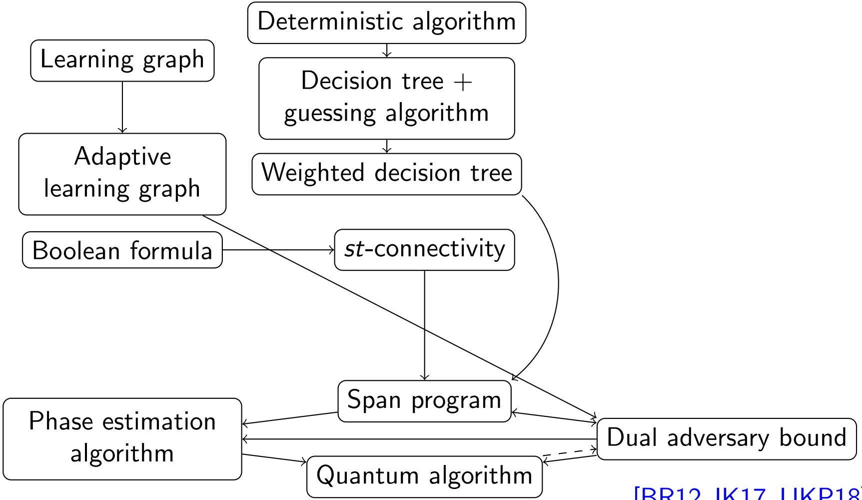

[BR12,JK17,JJKP18]

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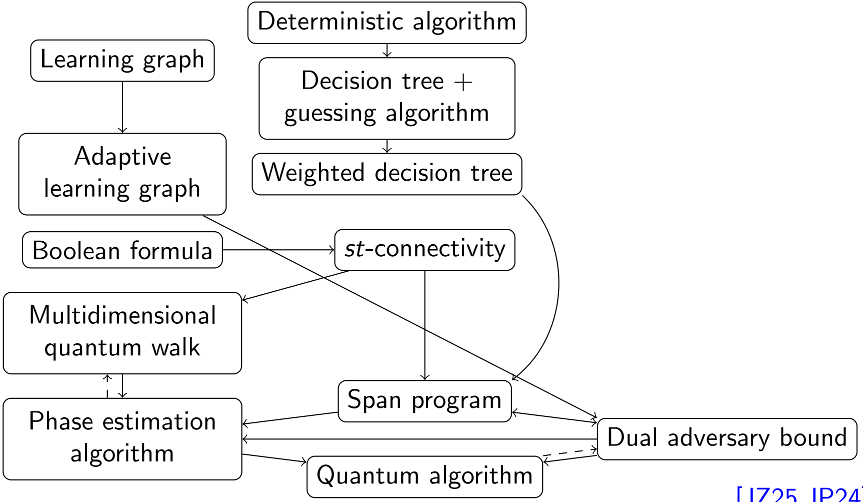

[JZ25,JP24]

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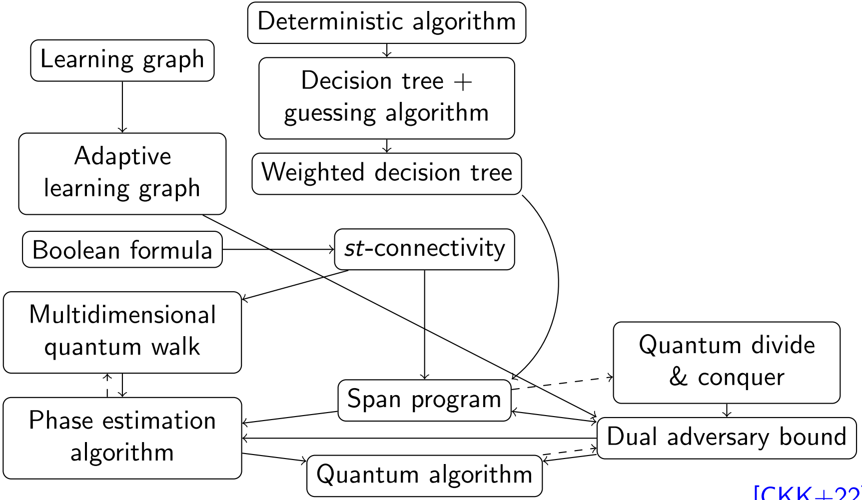

[CKK+22]

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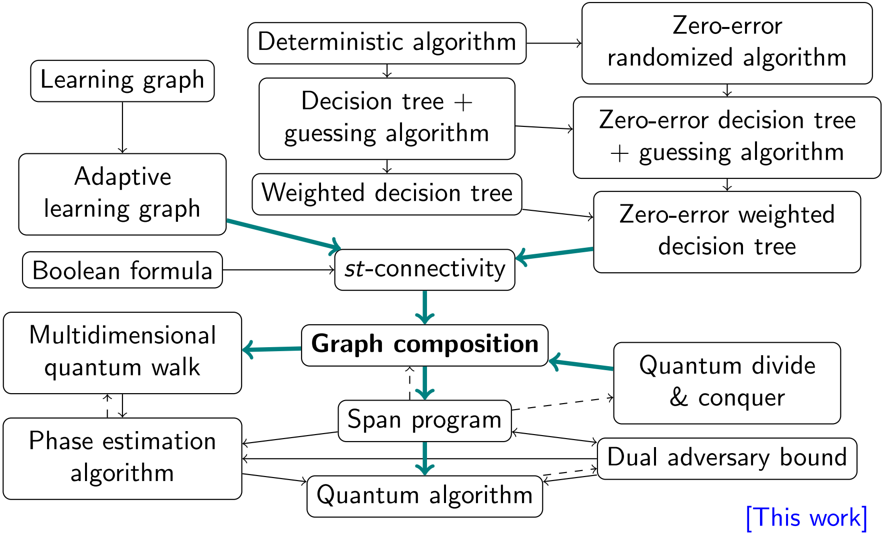

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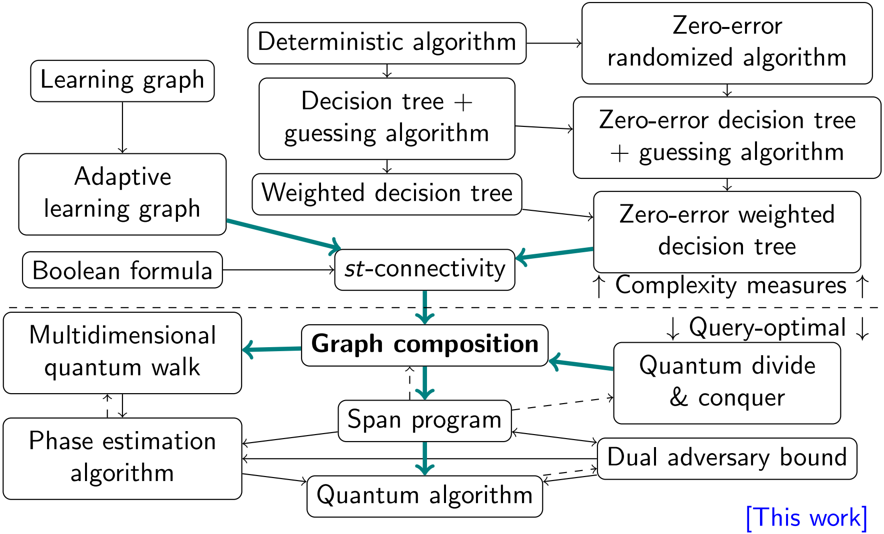

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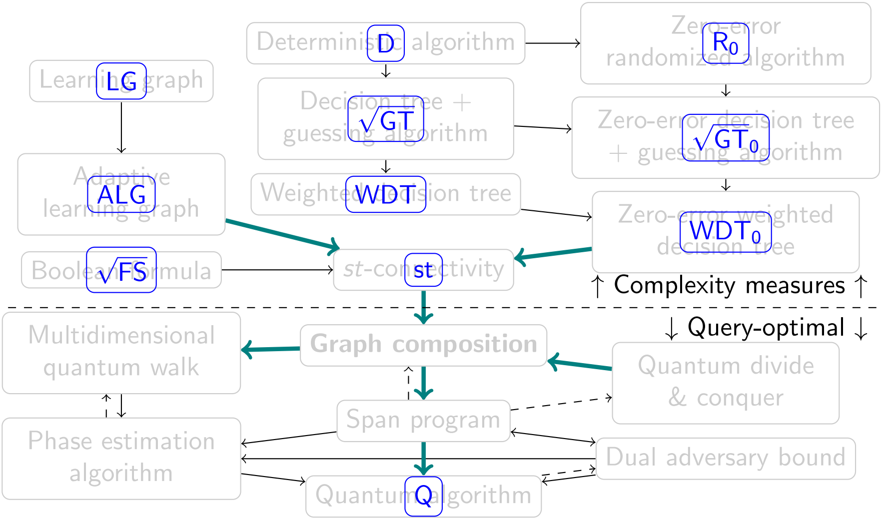
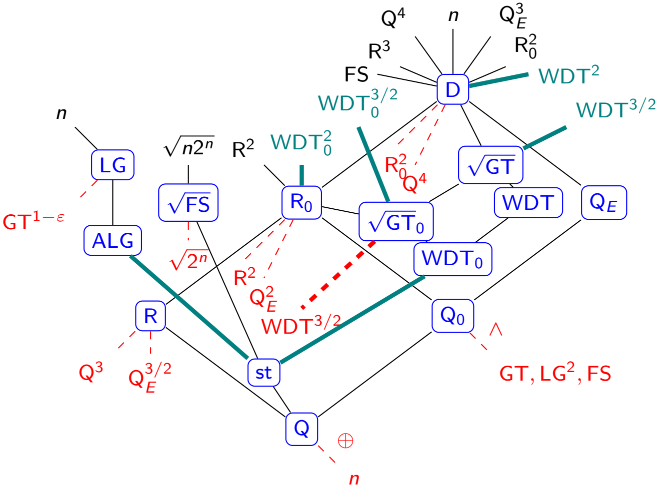
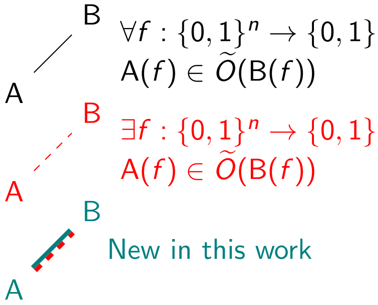


Complexity measure relations for total boolean functions



Legend:



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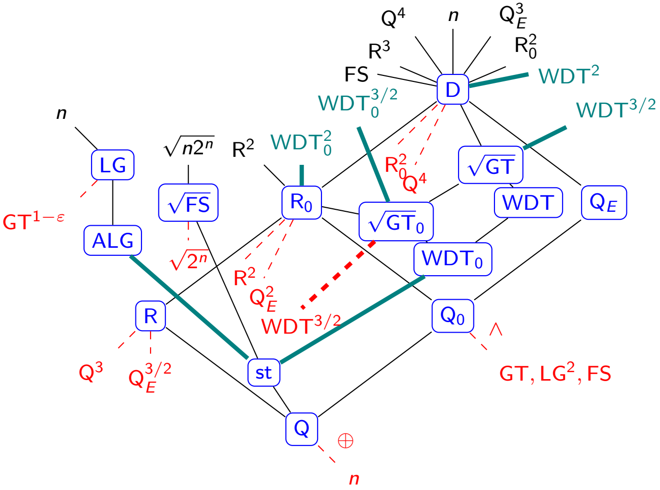

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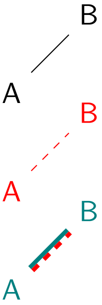

$\forall f : \{0,1\}^n \to \{0,1\}$
$\mathrm{A}(f) \in \widetilde{O}(\mathrm{B}(f))$

$\exists f : \{0,1\}^n \to \{0,1\}$
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New in this work

Open questions:

1. Separation between Q and st?
2. Can we prove $\mathrm{D} \in \widetilde{O}(\mathrm{st}^2)$?

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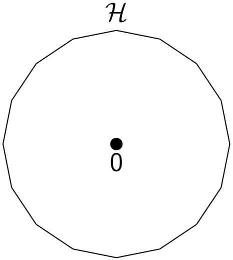

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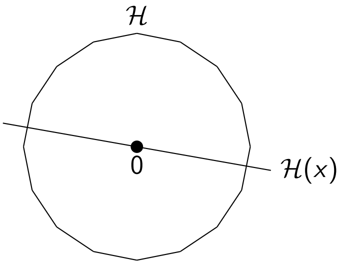

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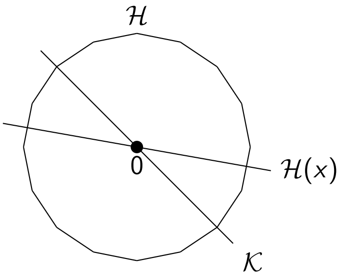

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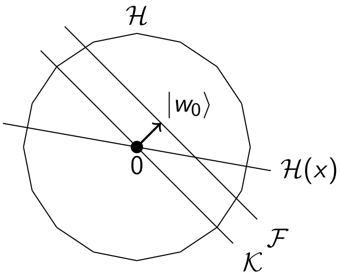

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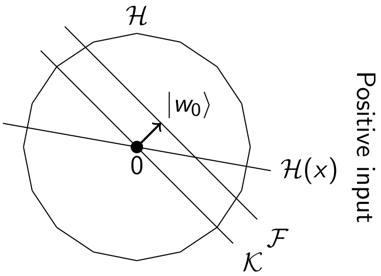

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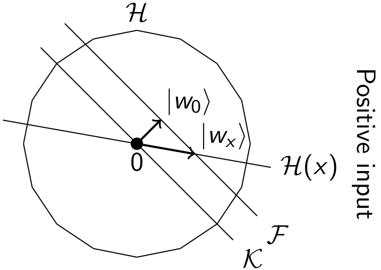

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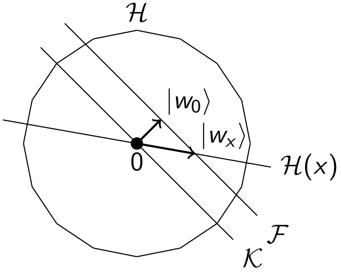
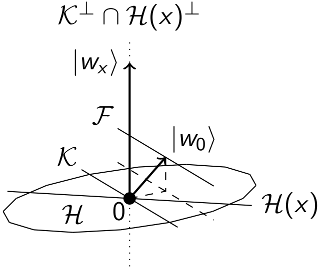


Positive input



Negative input

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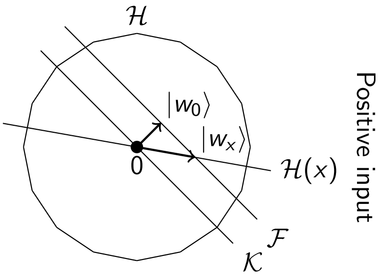
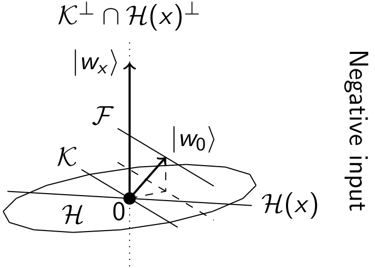
4. $C(\mathcal{P}) = \sqrt{\max_{x \in f^{-1}(0)} w_-(x, \mathcal{P}) \cdot \max_{x \in f^{-1}(1)} w_+(x, \mathcal{P})}$.





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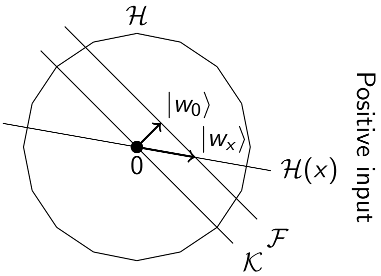
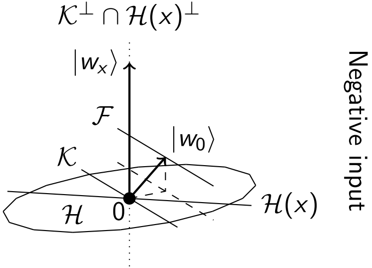
Thm: $Q(f; 2\Pi_{\mathcal{H}(x)} - I) = O(C(\mathcal{P}))$ [Rei11].





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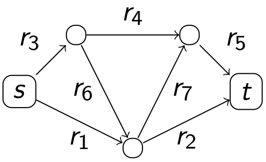
$G =$ 

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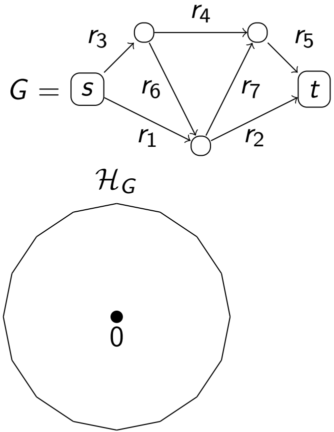

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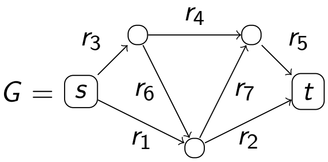

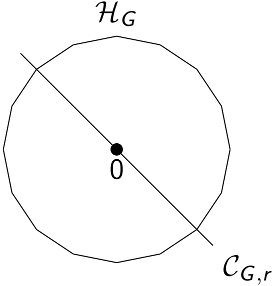

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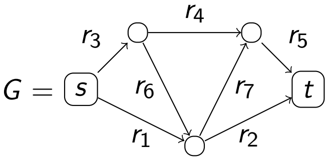

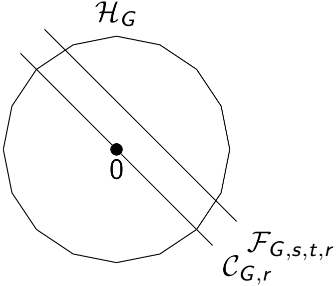

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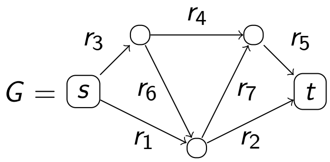
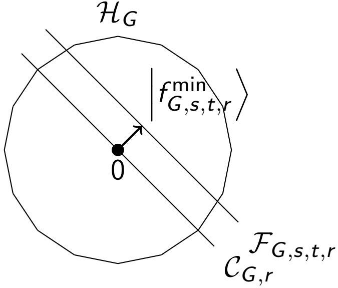
4. *Effective resistance:* $R_{G,s,t,r} := \||f^{\min}_{G,s,t,r}\rangle\|^2$.





Electrical networks and span programs [BR12, JK17, JJKP18]

Graph $G = (V, E)$, resistances $r : E \to [0, \infty]$, $s, t \in V$.

1. *Flow:* $f : E \to \mathbb{C}$.
   Flow space: $\mathcal{H}_G = \mathrm{Span}\{|e\rangle : e \in E\}$,
   $f \mapsto |f_{G,r}\rangle = \sum_{e \in E} f_e \sqrt{r_e}\, |e\rangle$.
2. *Circulation:* flow f with $\forall v \in V$,
   $\sum_{v \in N^+(v)} f_e - \sum_{v \in N^-(v)} f_e = 0$.
   Circulation space: $\mathcal{C}_{G,r} \subseteq \mathcal{H}_G$.
3. *Unit st-flow:* flow f with $\forall v \in V$,
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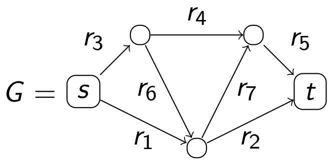
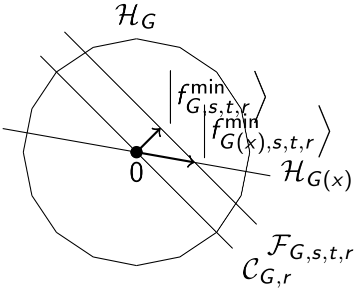
5. *Subgraph:* $x \in \{0,1\}^E \mapsto G(x) \mapsto \mathcal{H}_{G(x)} \subseteq \mathcal{H}_G$.





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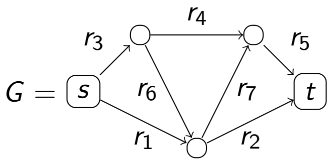

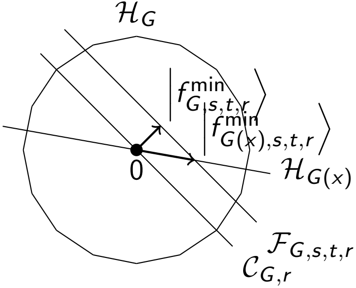

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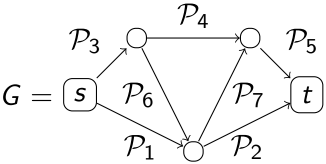

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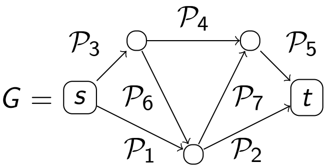

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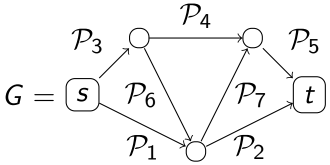

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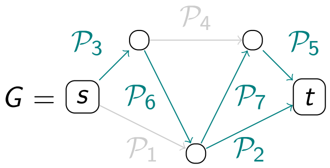

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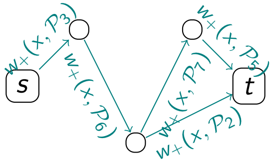

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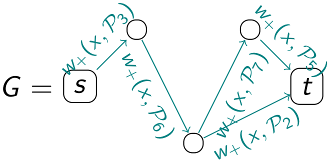

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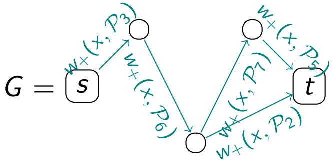

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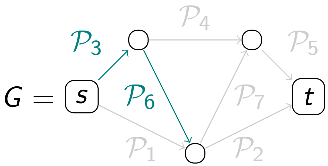

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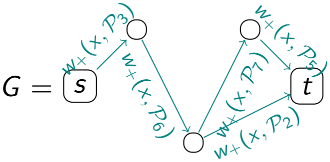

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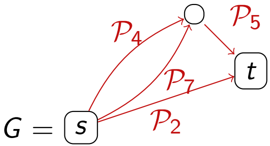

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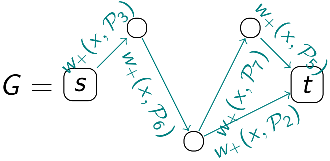

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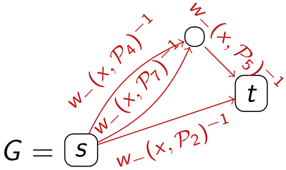

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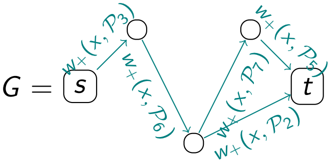

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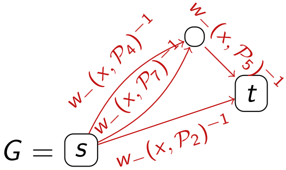

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Path-cut theorem

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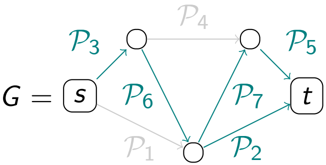

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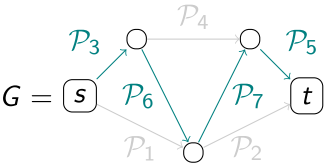

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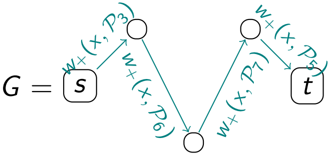

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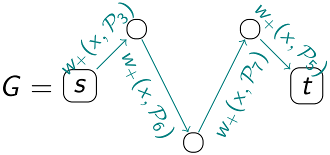

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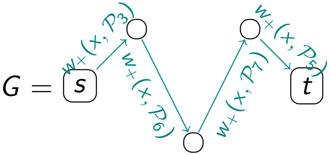

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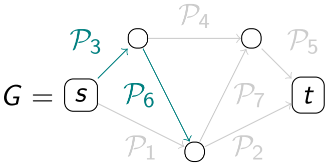

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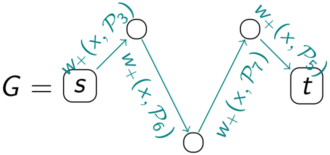

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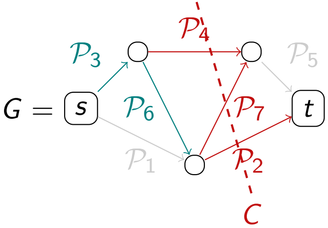

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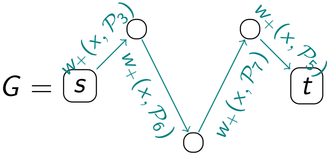

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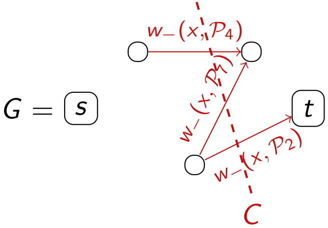

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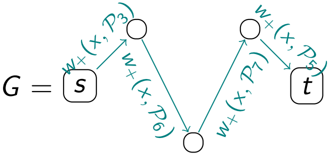

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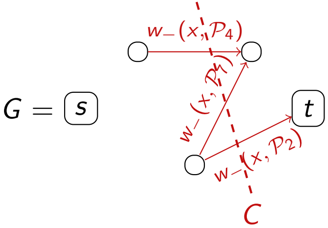

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Properties:

1. Simpler (less-powerful) version.
2. Still powerful enough for many applications.

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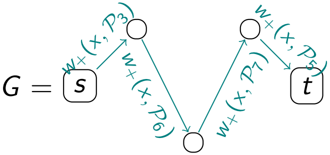

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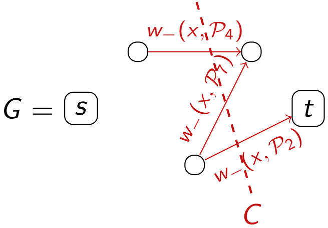

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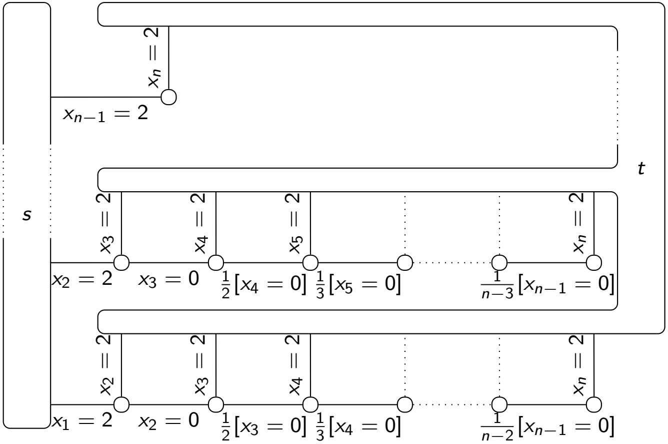

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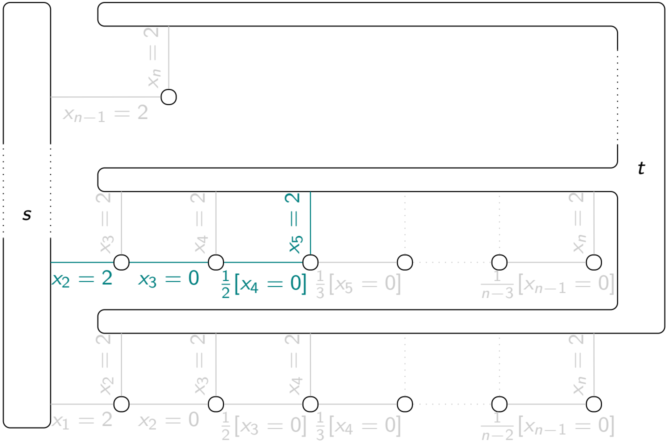

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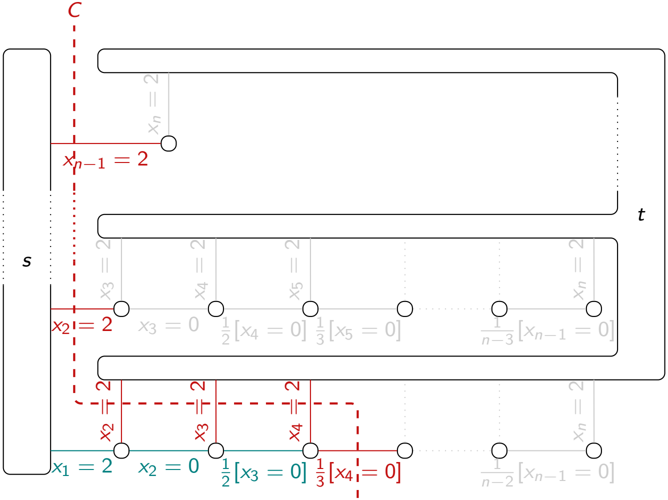

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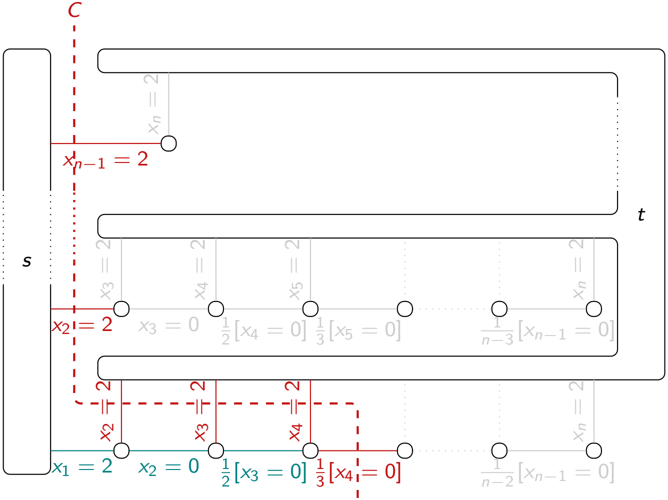

(Time-efficient) Implementation (I/II)

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Three operations: (each is called $O(C(\mathcal{P}))$ times)

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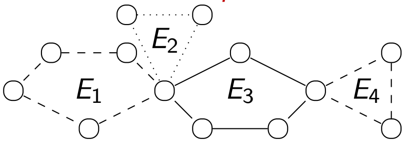

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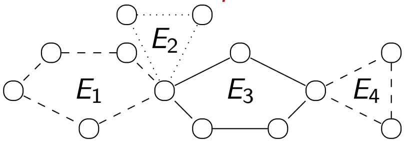

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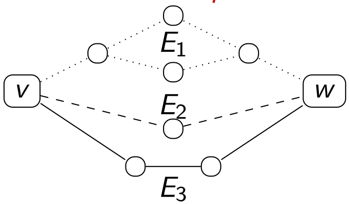

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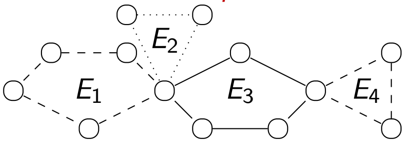
3. Always possible to decompose.

Tree decomposition:



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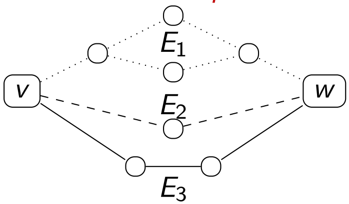

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    1. Every leaf is a single edge, i.e., $E_{j_1,\dots,j_d} = \{e\}$.

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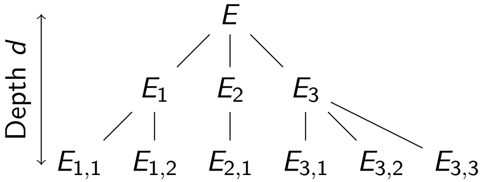

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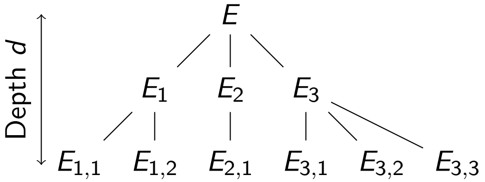

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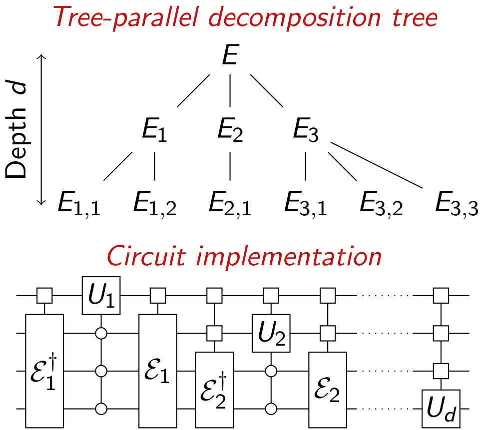

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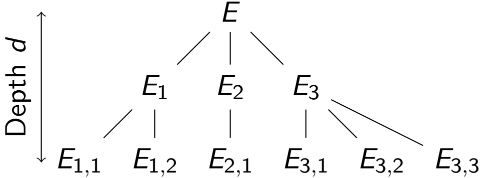

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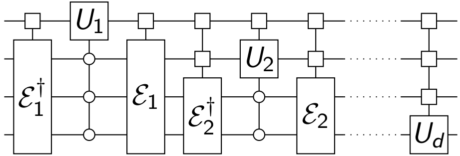

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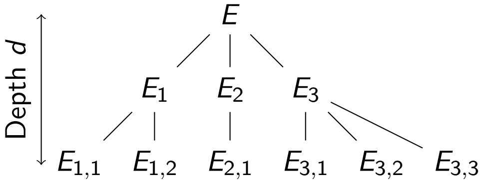

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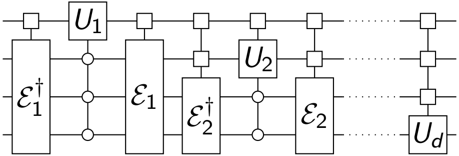

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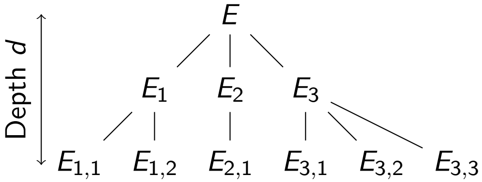

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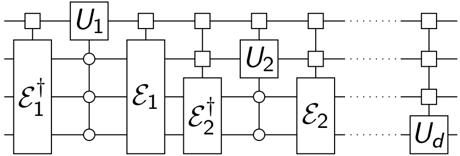

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   4. *Total cost:*
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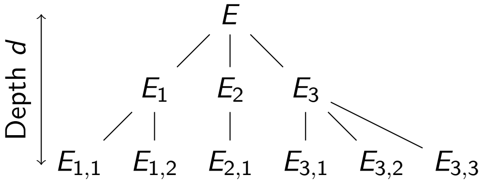

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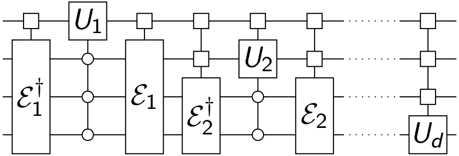

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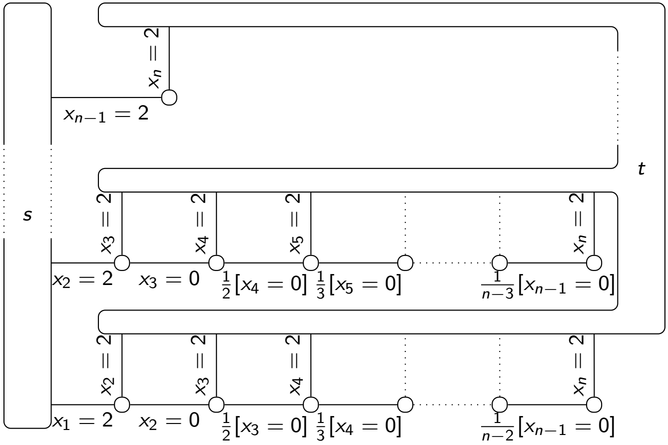

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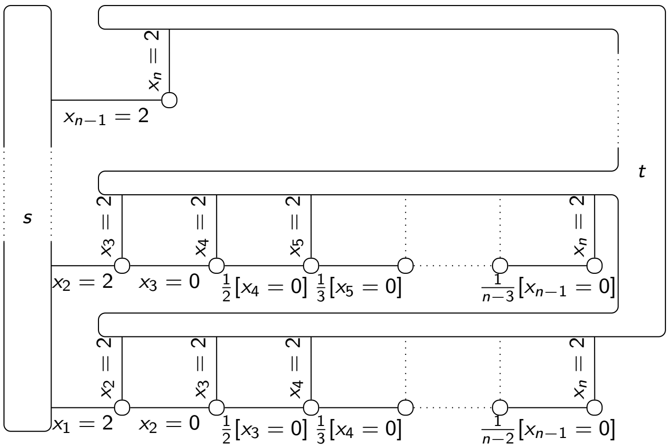

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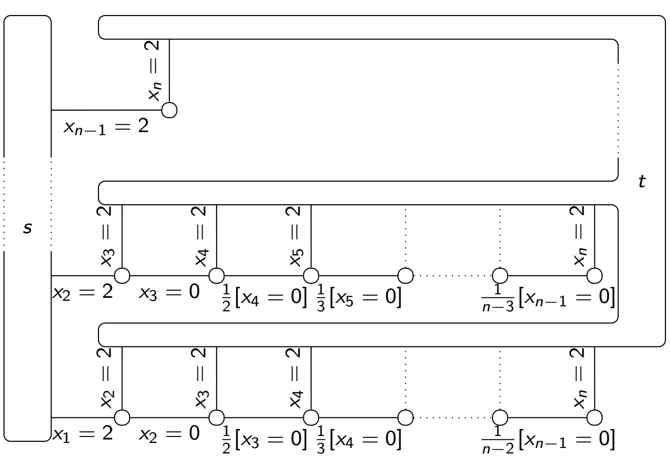

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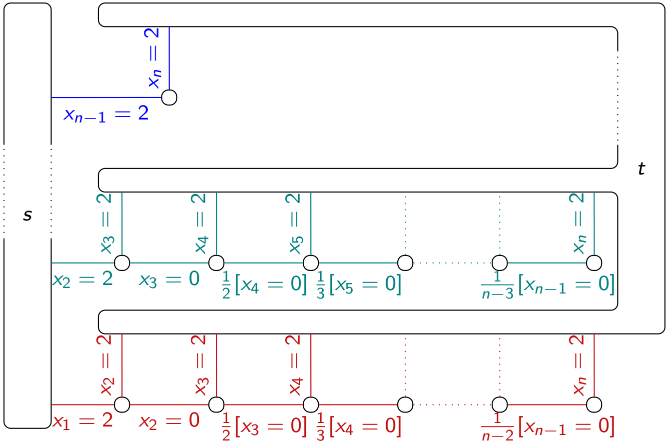

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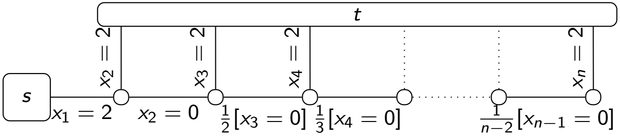

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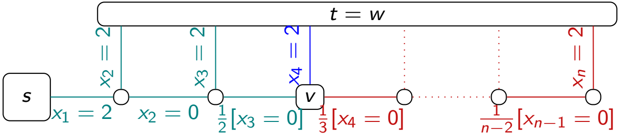

Example: Time-efficient implementation of the $\Sigma^*20^*2\Sigma^*$-problem

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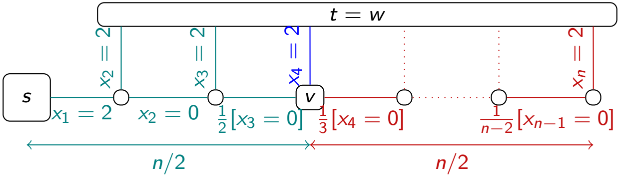

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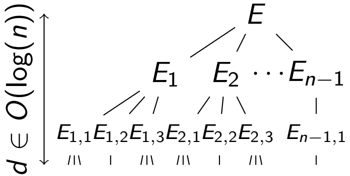

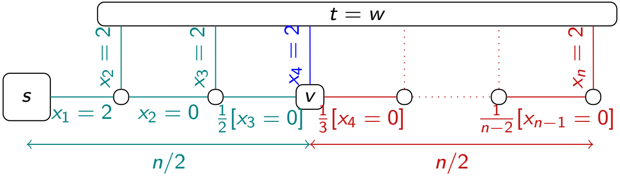

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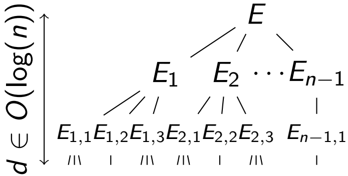

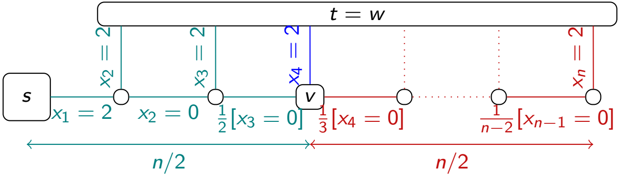

Total cost:

1. $O(\sqrt{n\log(n)})$ queries.
2. $\widetilde{O}(\sqrt{n})$ time.
3. $\widetilde{O}(n^2)$ bits of QROM. (Further ad-hoc improvements possible).

Summary (I/II)

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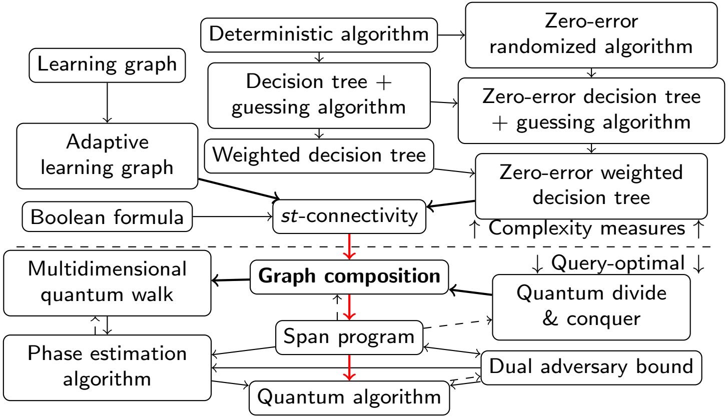
Relations between algorithmic frameworks



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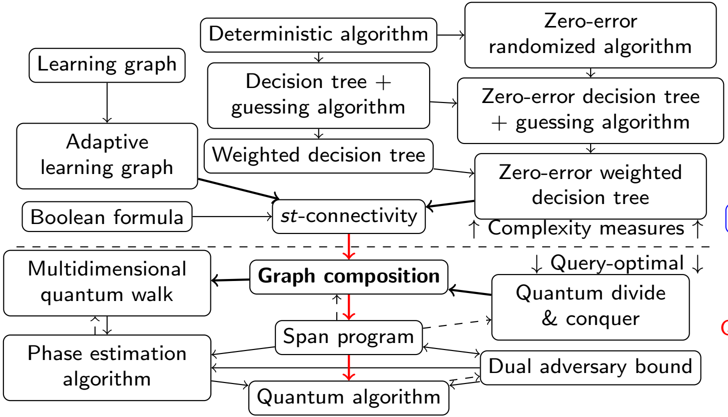
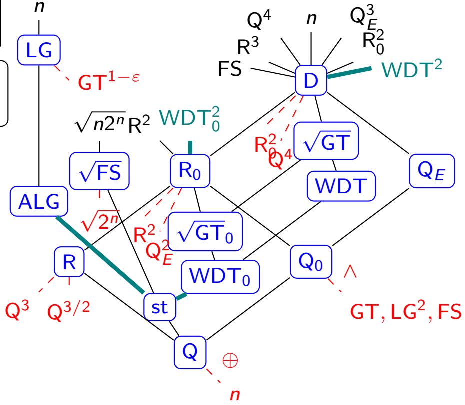


Relations between complexity measures



Summary (II/II)

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Thanks for your attention!

`ajcornelissen@outlook.com`

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