How to estimate the volume in low dimension?

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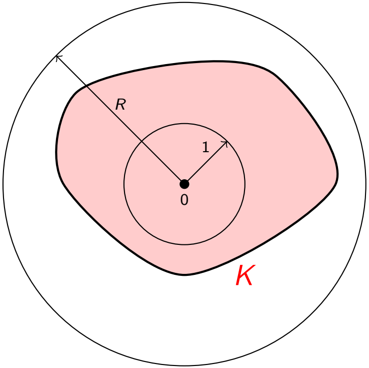

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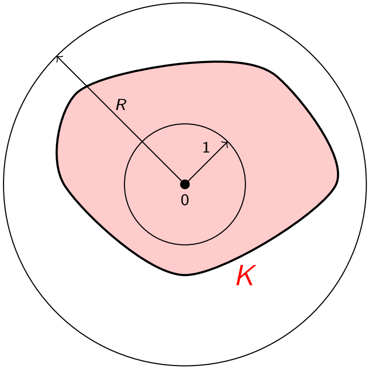

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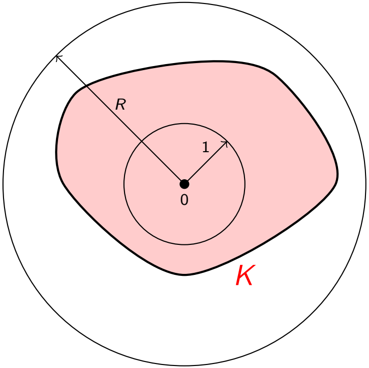

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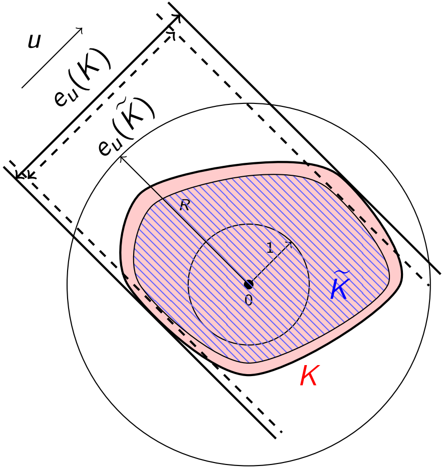

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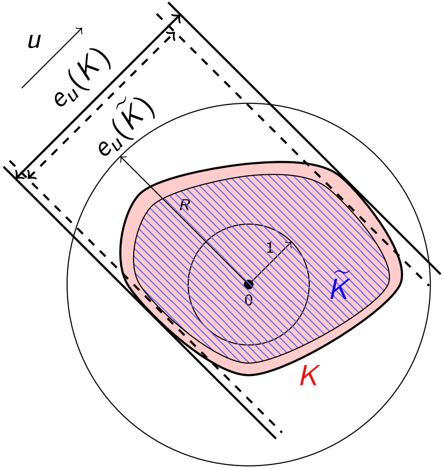
    2. *Volume estimation:*
    Output $\widetilde{V} \geq 0$ such that $\frac{|\widetilde{V} - \mathrm{Vol}(K)|}{\mathrm{Vol}(K)} \leq \varepsilon$.



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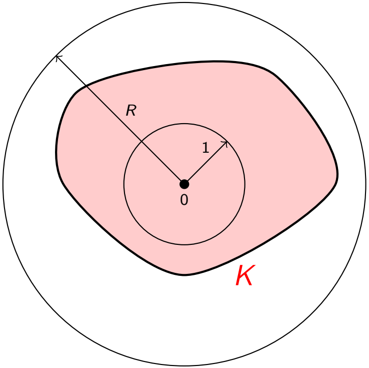

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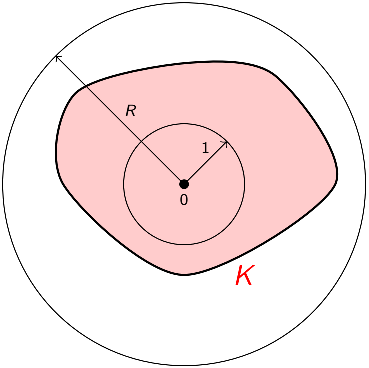

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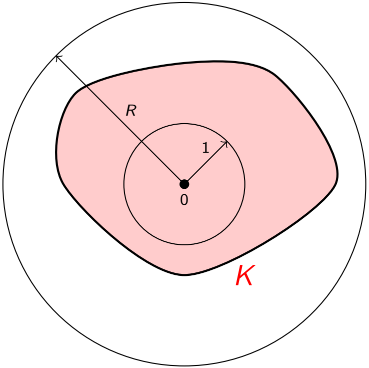

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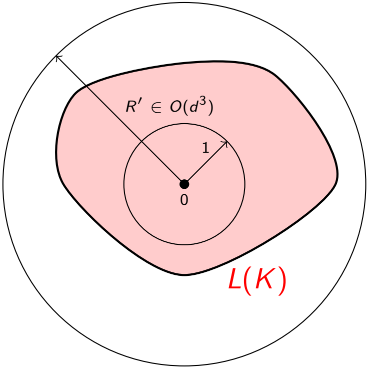
4. *Rounding:* [GLS88]
   1. Finds an affine transformation: $L : x \mapsto x_0 + Tx$,
   2. Such that $B_d \subseteq L(K) \subseteq R' B_d$ with $R' \in O(d^3)$,
   3. In $\widetilde{O}(\text{poly}(d))$ deterministic queries.



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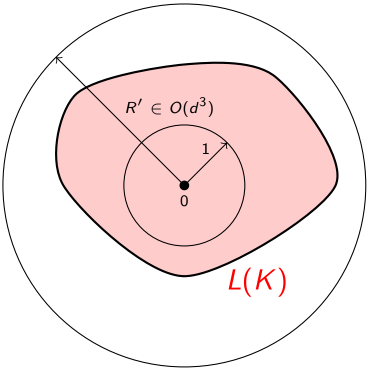

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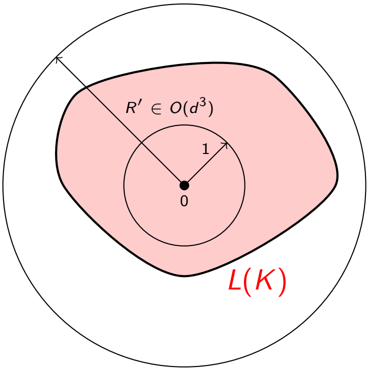
5. *Observation:* No polynomial dependence on R.



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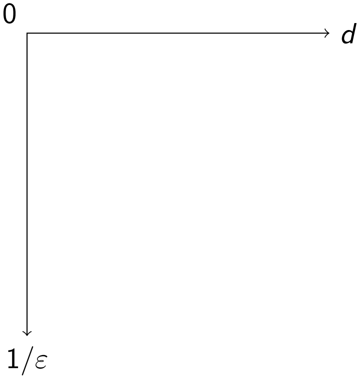

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    1. [DF91]: Randomized – $\widetilde{O}(\text{poly}(d, 1/\varepsilon))$
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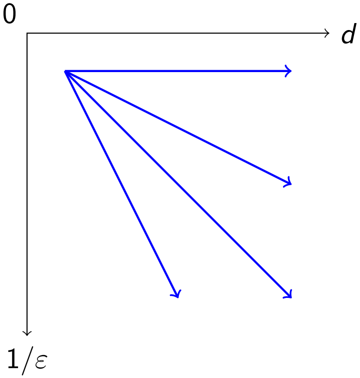

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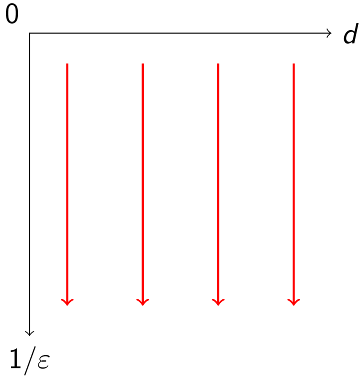

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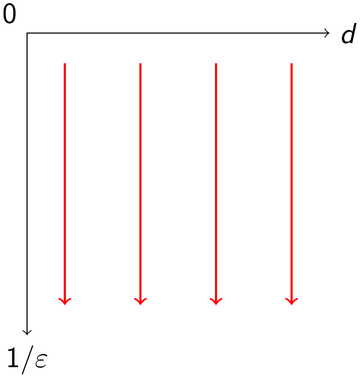
   1. [Chan06]: $K = \text{conv}(\{x_1, \dots, x_n\})$:
      Deterministic time complexity:
      1. $O(n + \varepsilon^{-1/2})$ for $d = 2$,
      2. $\widetilde{O}(n + \varepsilon^{-(d-2)})$ for $d \geq 3$.



Naive algorithms

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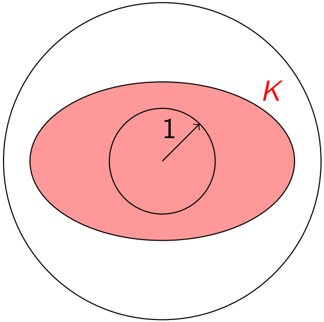

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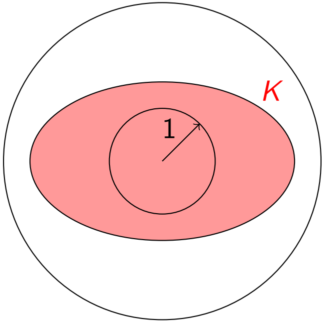

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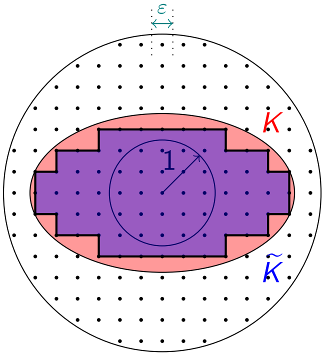

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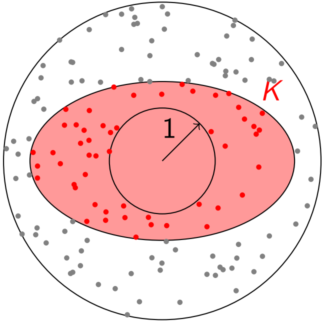

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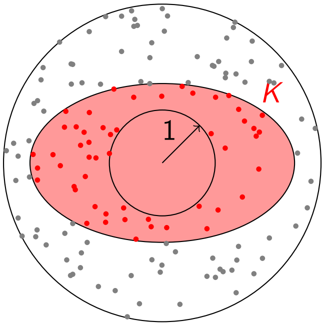

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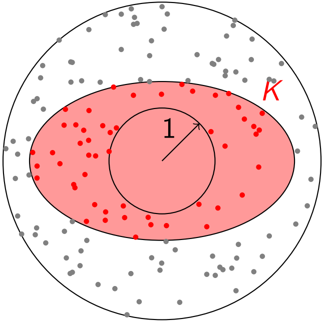
3. *Naive lower bounds:* $\Omega(1)$.



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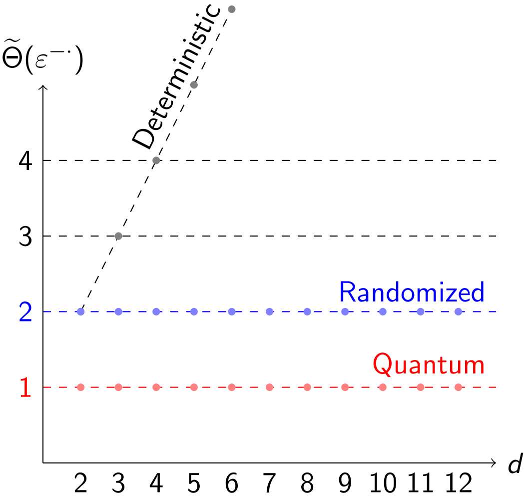

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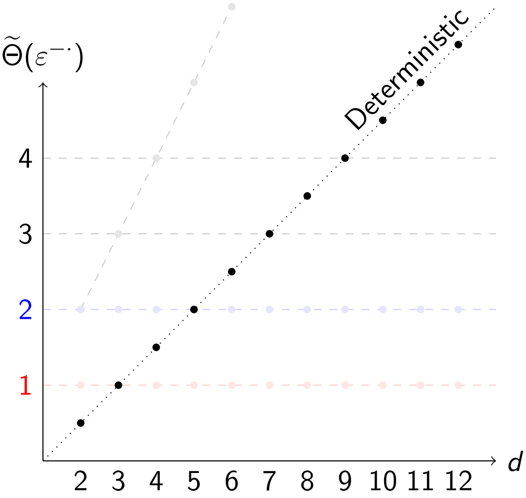

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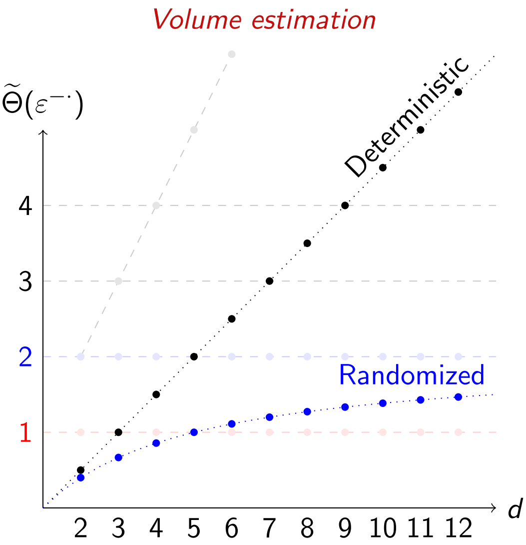

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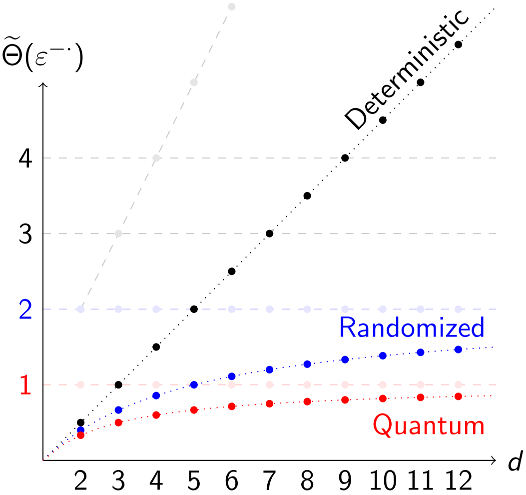

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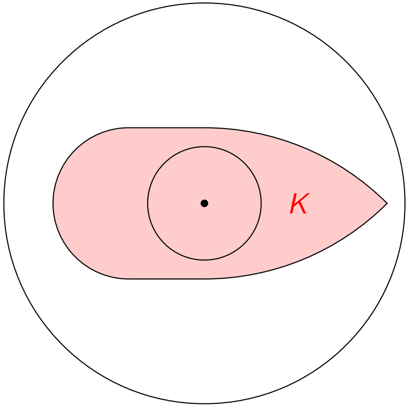

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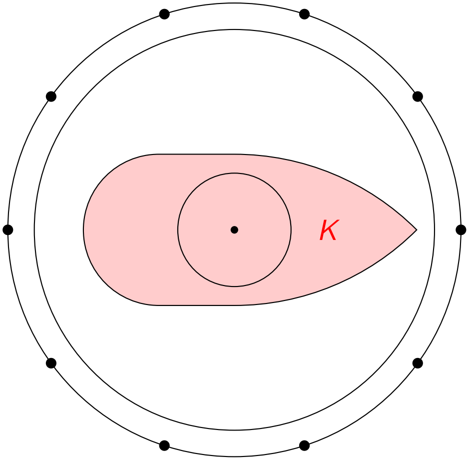

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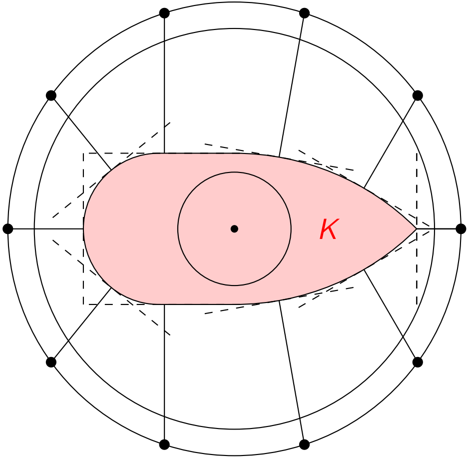

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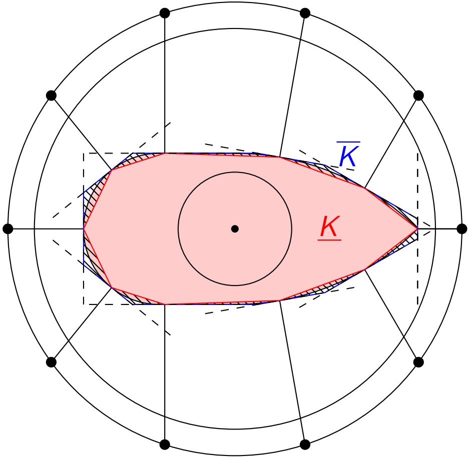
   3. Define $\underline{K}$, $\overline{K}$ as in the picture.



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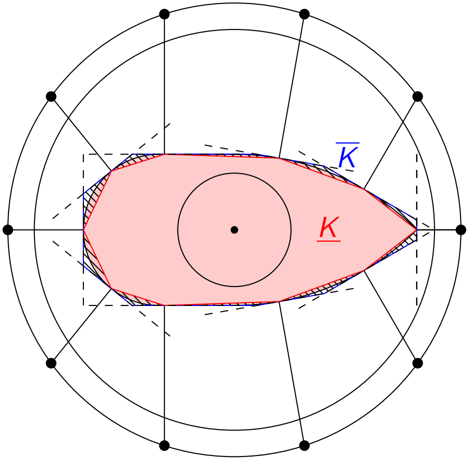

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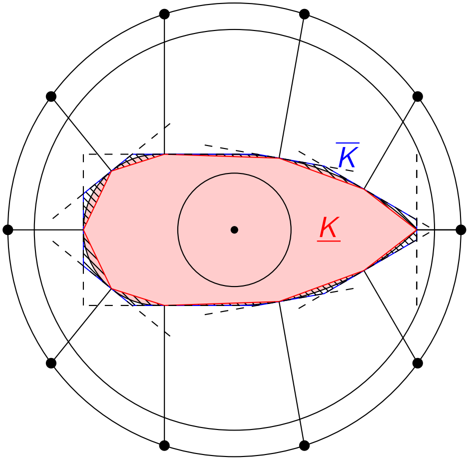

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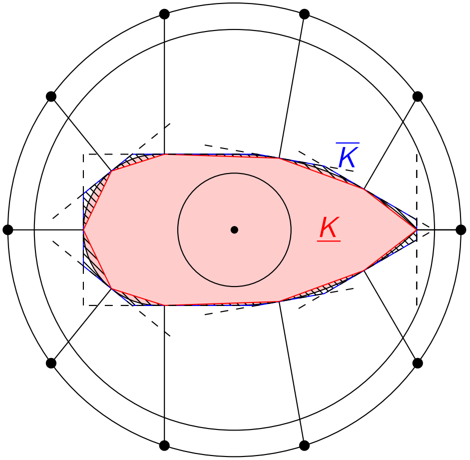

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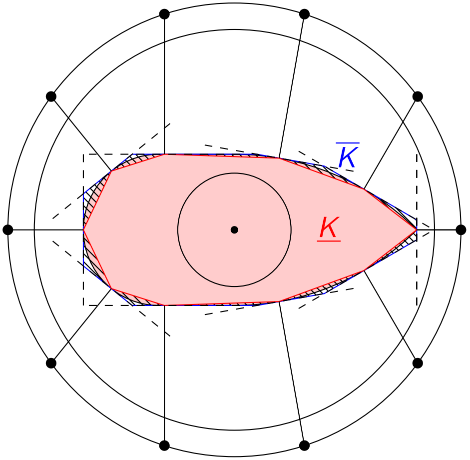

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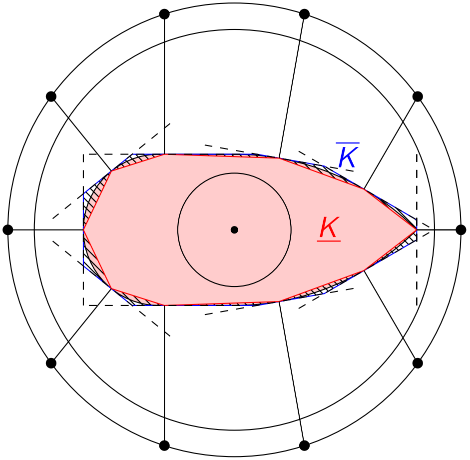

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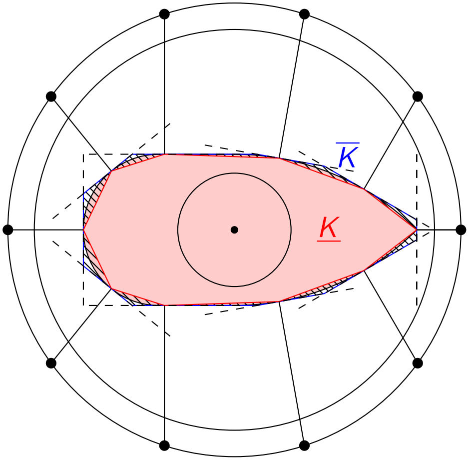
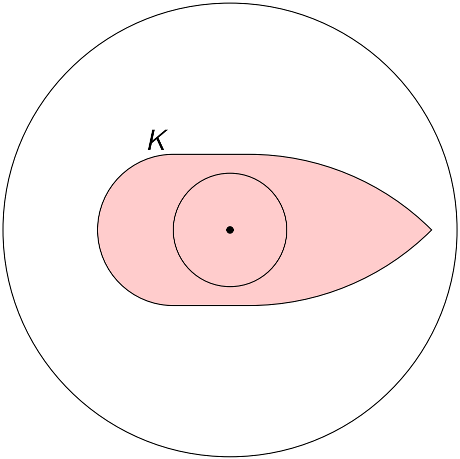
5. *Remark:* Approximation errors. [Chan06]



Algorithm II – Volume estimation



Algorithm II – Volume estimation

1. *Procedure:* $(\delta > 0)$
    1. Find a δ-kernel deterministically:
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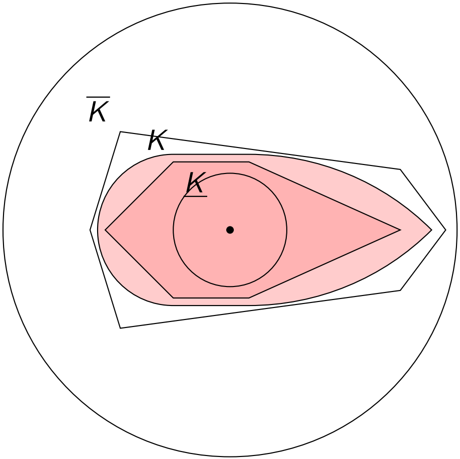

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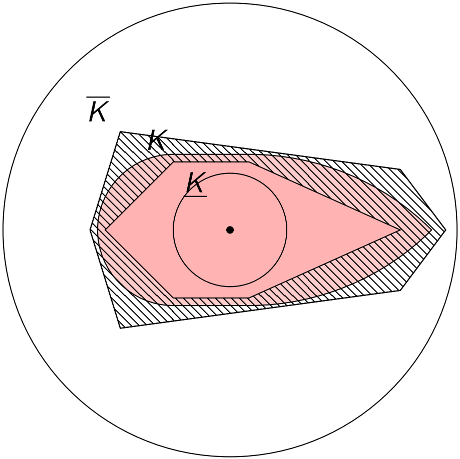

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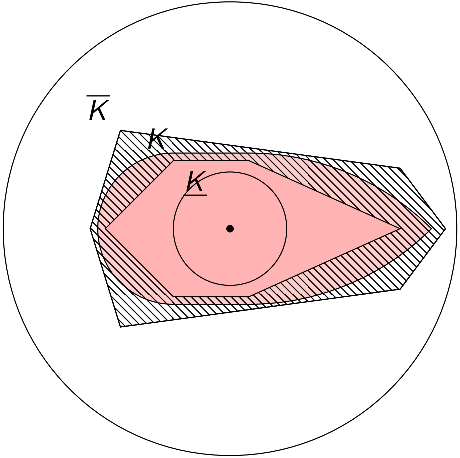

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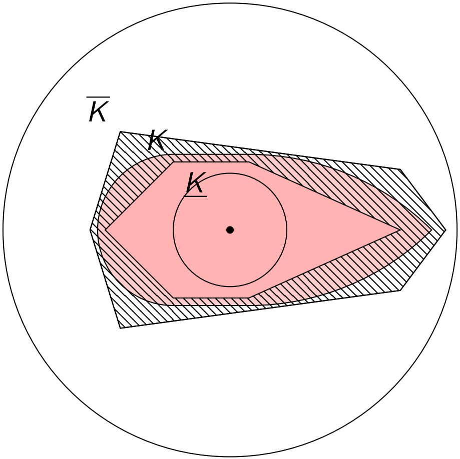
3. *Balance:* Optimize δ.

   $\Rightarrow$ All complexities follow.



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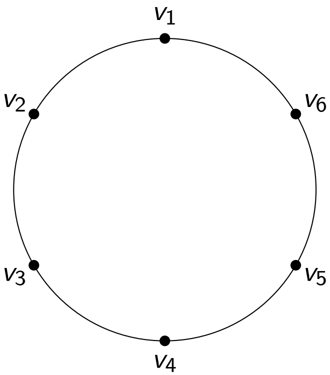

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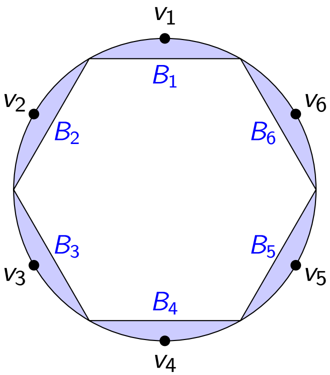

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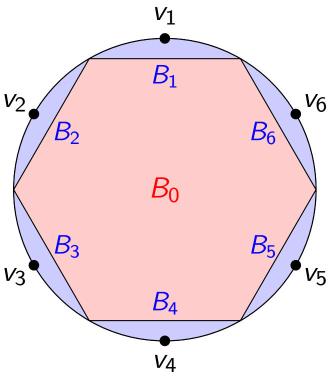

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Model	Recovery $\lvert x \oplus \tilde{x}\rvert \leq k$	Approx. counting $\lvert\lvert x\rvert - \tilde{w}\rvert \leq k$
Deterministic	$\Theta(n)$	$\Theta(n)$
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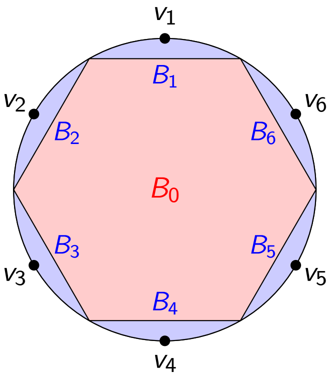

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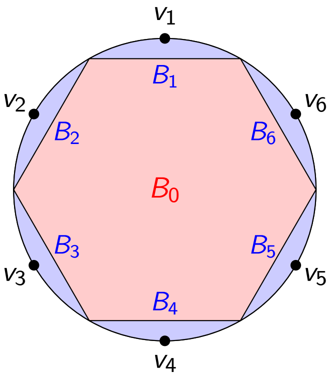

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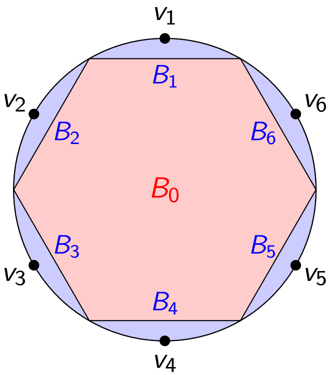

Summary

Our results: ($d \in \mathbb{N}$ fixed, $\varepsilon \downarrow 0$)

Model	Kernel est.	Volume est.
Deterministic	$\widetilde{\Theta}(\varepsilon^{-\frac{d-1}{2}})$	$\widetilde{\Theta}(\varepsilon^{-\frac{d-1}{2}})$
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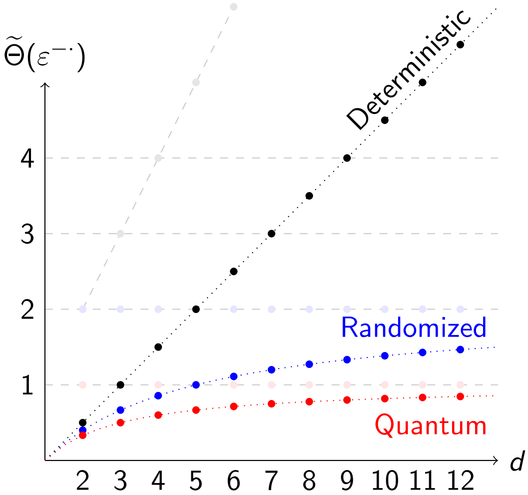

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Thanks for your attention!

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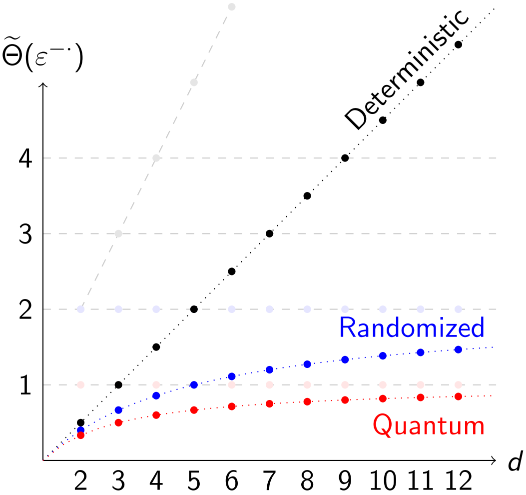
Volume estimation



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