Span programs and quantum time complexity

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UNIVERSITY OF AMSTERDAM

QuSoft
Research Center for Quantum Software

CWI

Overview

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1. High-level discussion

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2. Technical part

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   1. Span program $\Rightarrow$ quantum algorithm
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   1. Span program $\Rightarrow$ quantum algorithm
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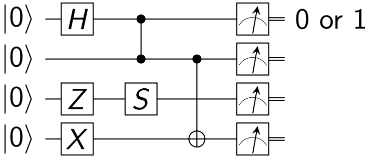
3. Application to variable-time search

Quantum query algorithms

Quantum query algorithms



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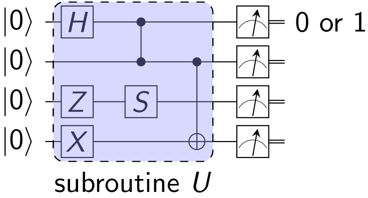

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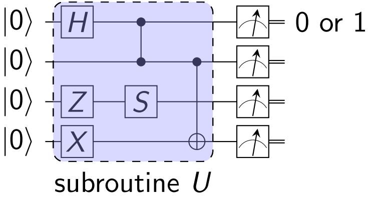

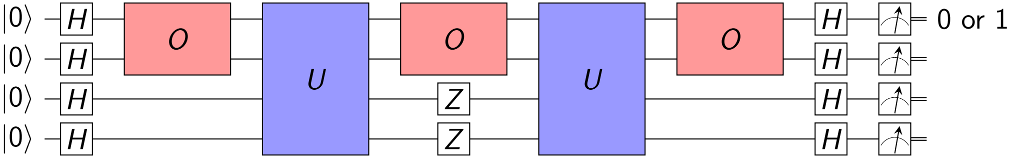

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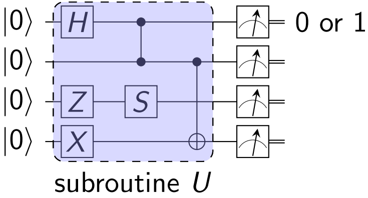

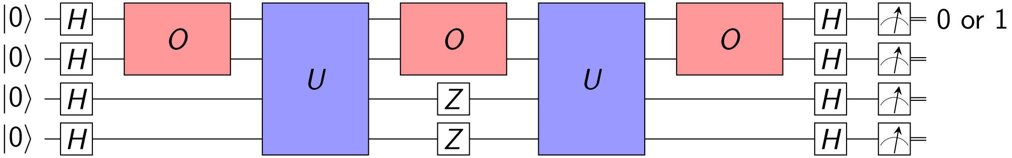

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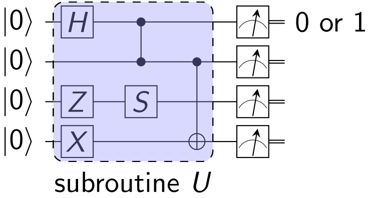

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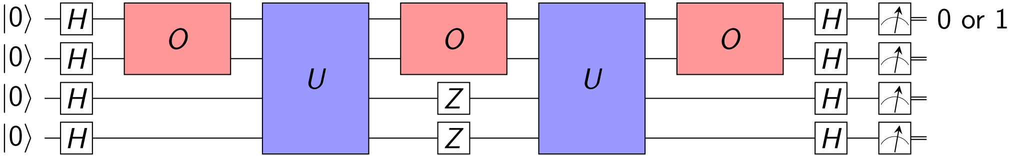

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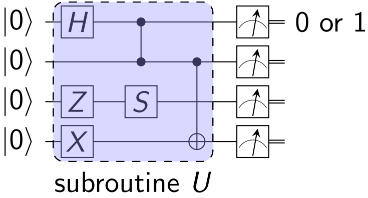

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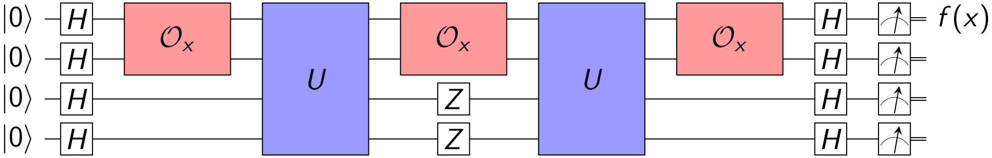

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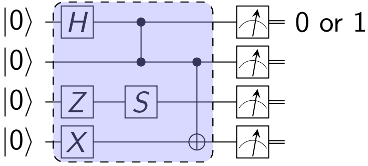

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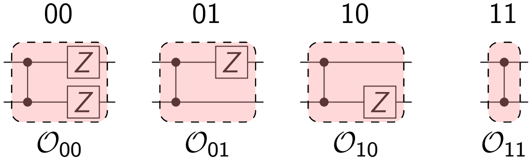

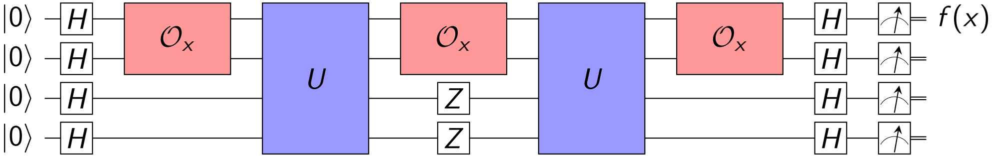

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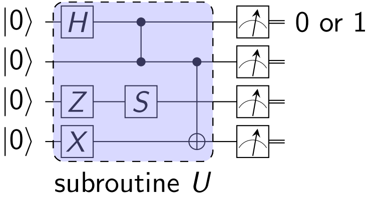

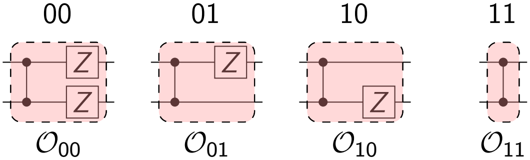

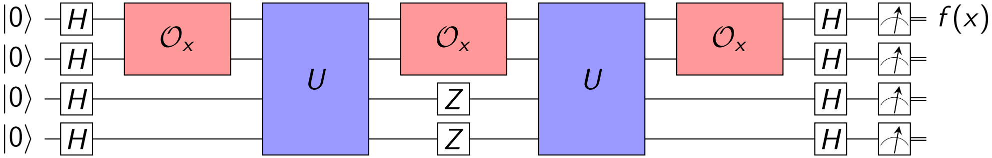

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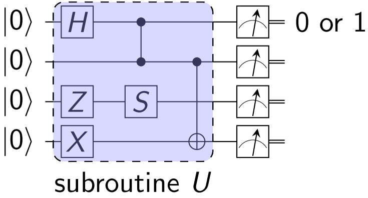

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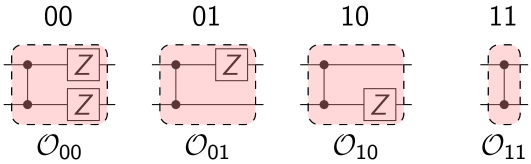

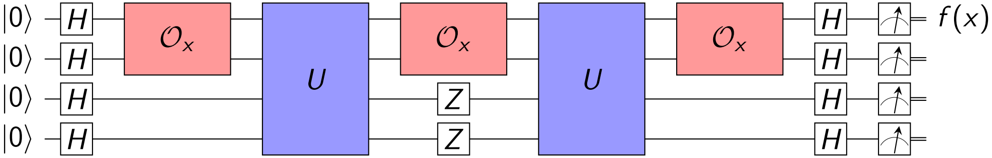

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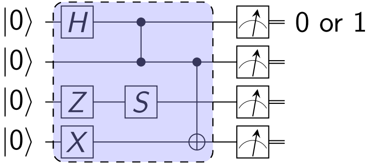

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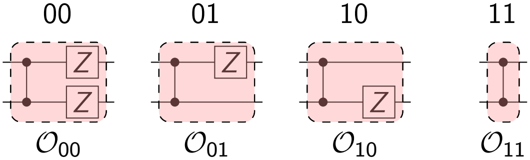

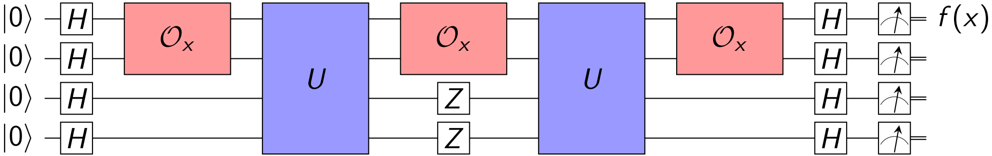

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k:	Space complexity	no. qubits	$k = 4$

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1. Encode f into several mathematical objects.
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Motivation 1: We wish to learn more about the time complexity of span program algorithms.

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Corollary: For every f, there exists a span program that generates a quantum algorithm that computes f with optimal query, time and space complexity.

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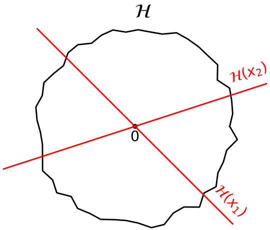

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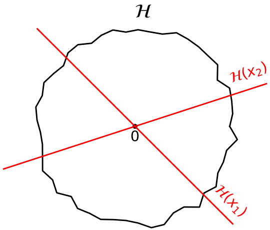

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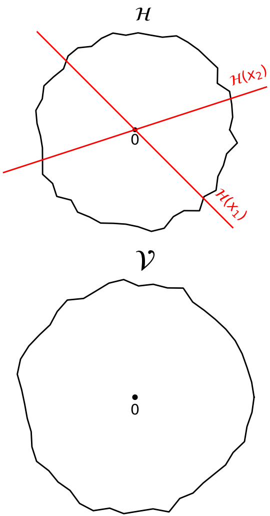

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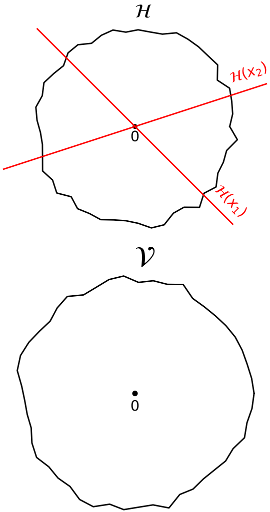

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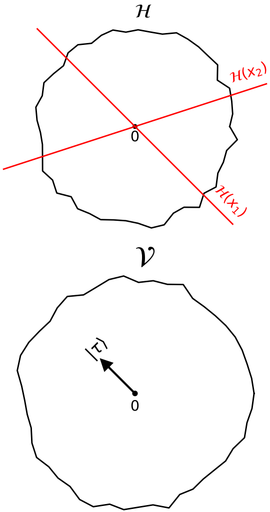

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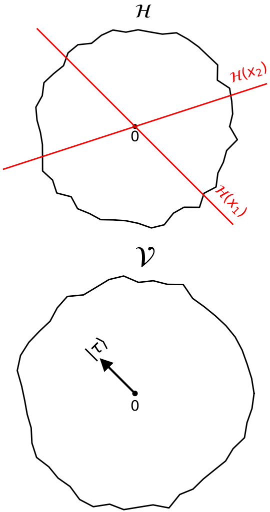

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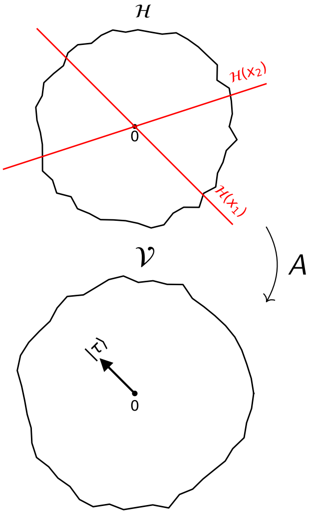

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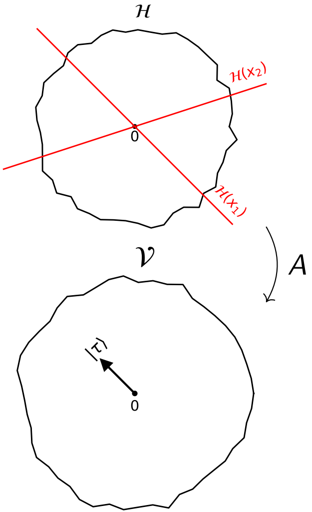

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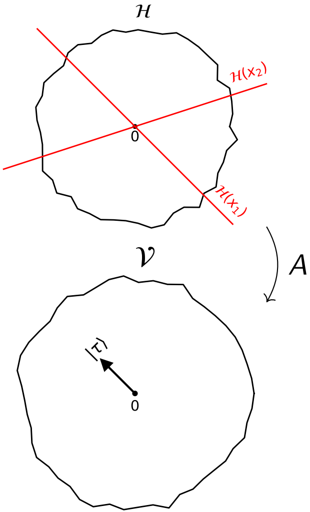
Span program evaluates f if
$f(x) = 1 \Leftrightarrow |w_0\rangle \in \mathcal{H}(x) + \ker(A)$.



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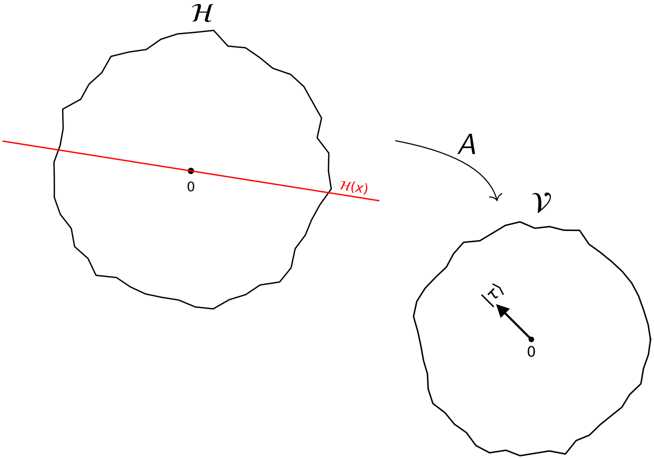

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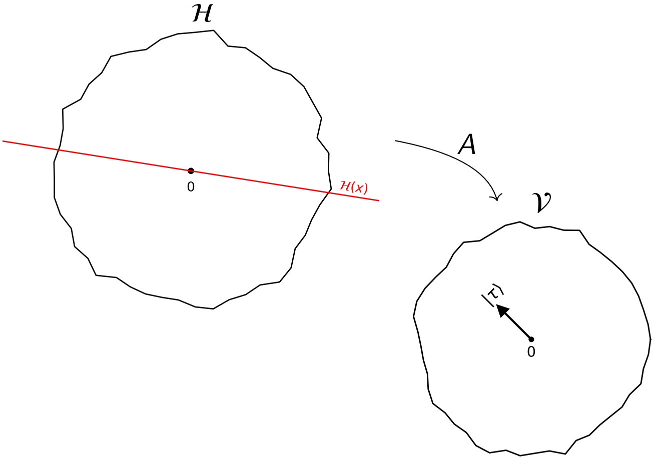
Positive instance: $f(x) = 1$



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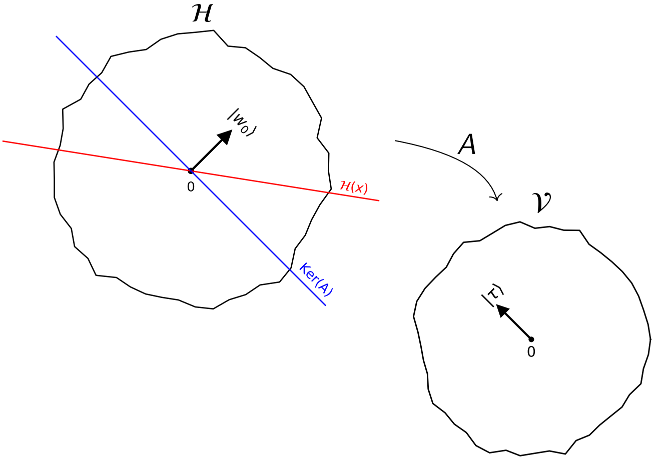

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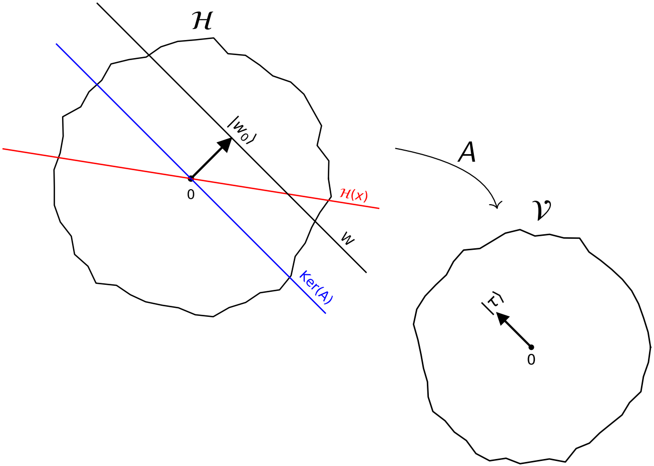

Span programs – visualization

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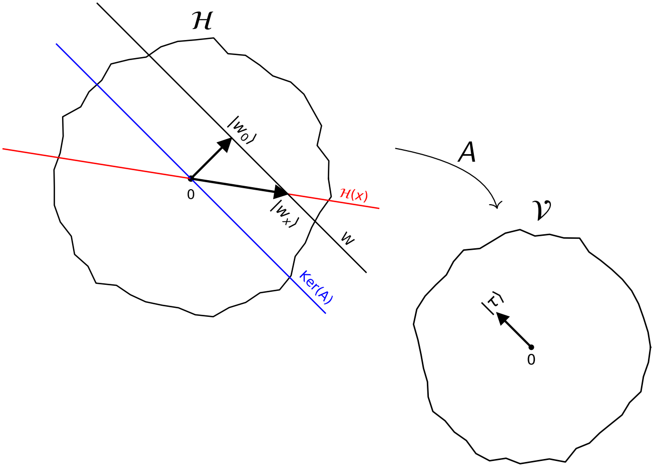

Span programs – visualization

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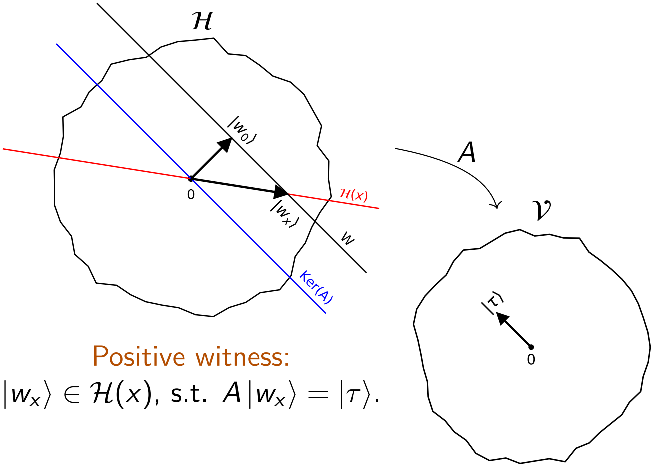
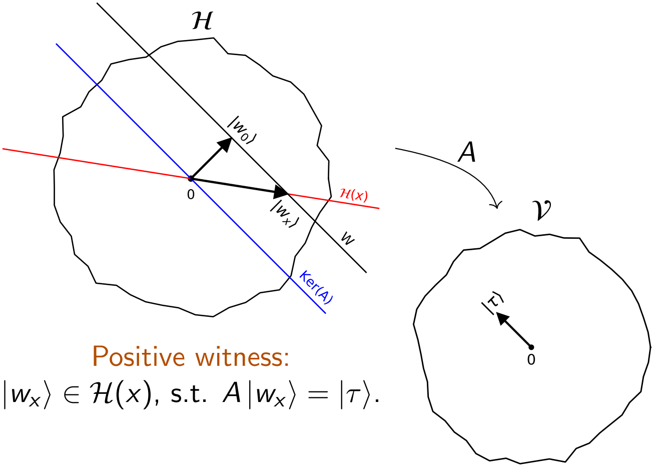


Positive witness:
$|w_x\rangle \in \mathcal{H}(x)$, s.t. $A|w_x\rangle = |\tau\rangle$.

Span programs – visualization

Positive instance: $f(x) = 1$

Negative instance: $f(x) = 0$



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Span programs – visualization

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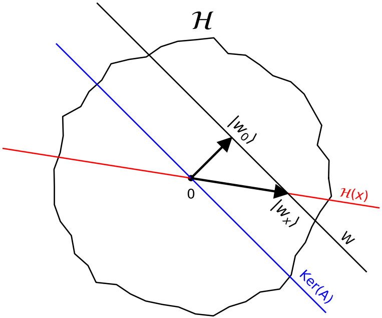

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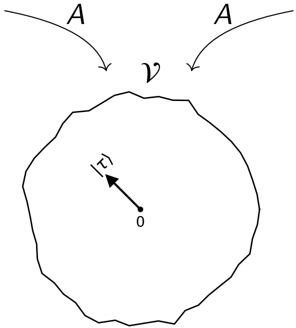
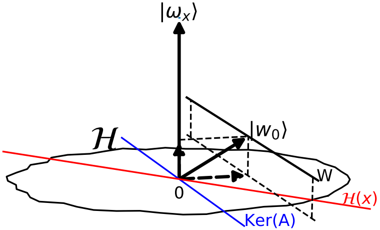


Negative instance: $f(x) = 0$



Span programs – visualization

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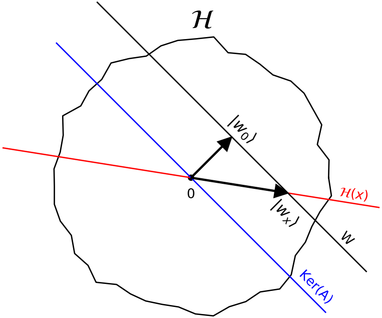

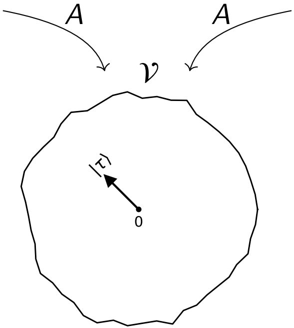

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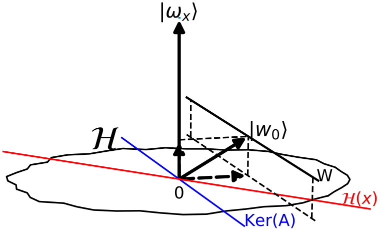


Positive witness:

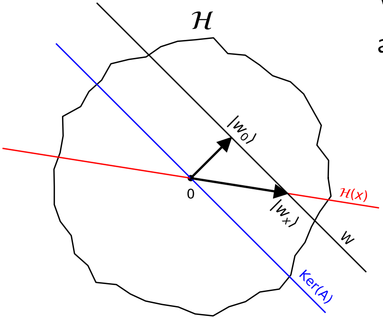
$|w_x\rangle \in \mathcal{H}(x)$, s.t. $A|w_x\rangle = |\tau\rangle$.

Negative witness:

$|\omega_x\rangle \in \mathcal{H}(x)^\perp \cap \ker(A)^\perp$, s.t. $\langle \omega_x | w_0 \rangle = 1$.

Span programs – visualization

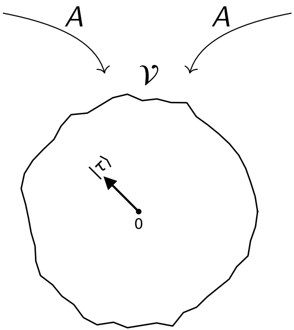
Positive instance: $f(x) = 1$



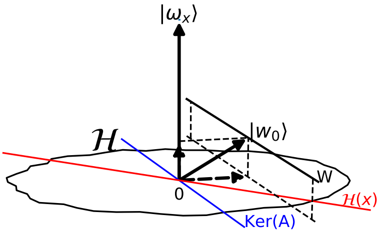
Positive witness:

$|w_x\rangle \in \mathcal{H}(x)$, s.t. $A|w_x\rangle = |\tau\rangle$.

We reflect through $\mathcal{H}(x)$ and then through $\ker(A)$.



Negative instance: $f(x) = 0$



Negative witness:

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Span programs – visualization

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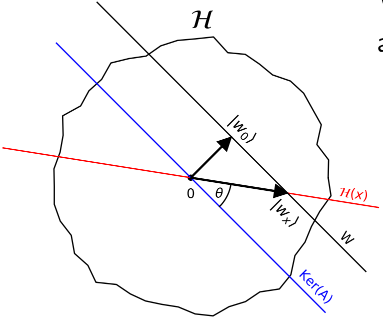

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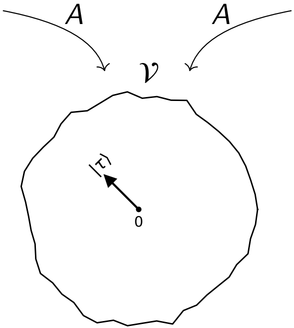

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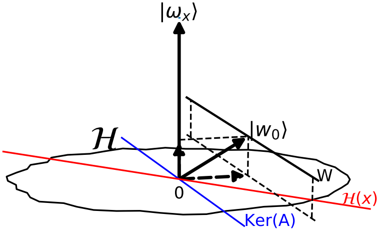

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Span programs – visualization

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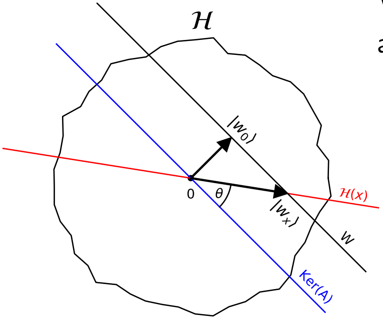
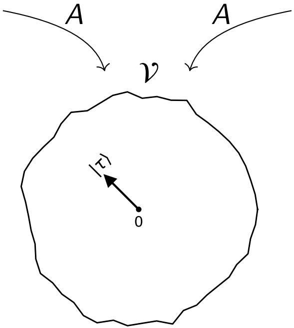
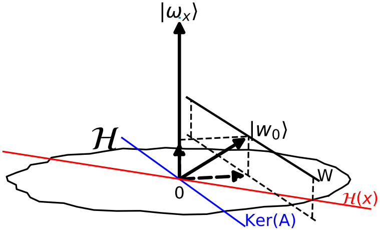


We reflect through $\mathcal{H}(x)$ and then through $\ker(A)$.



Negative instance: $f(x) = 0$



Positive witness:

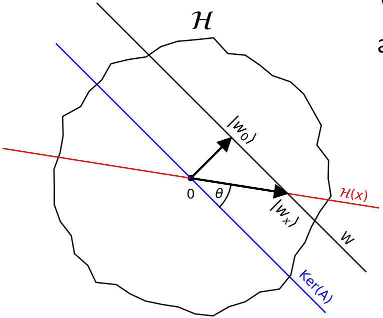
$|w_x\rangle \in \mathcal{H}(x)$, s.t. $A|w_x\rangle = |\tau\rangle$.
$|w_0\rangle$ rotates at angle 2θ,
$\theta \geq \sin\theta = \||w_0\rangle\| / \||w_x\rangle\|$.

Negative witness:

$|\omega_x\rangle \in \mathcal{H}(x)^\perp \cap \ker(A)^\perp$, s.t.
$\langle \omega_x | w_0 \rangle = 1$.

Span programs – visualization

Positive instance: $f(x) = 1$



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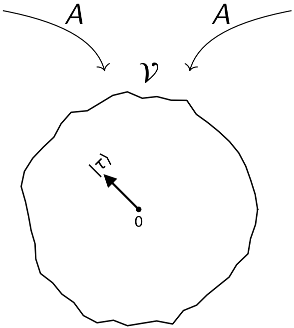

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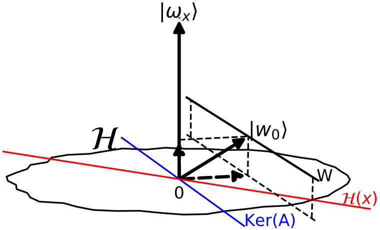

Negative witness:

$|\omega_x\rangle \in \mathcal{H}(x)^\perp \cap \ker(A)^\perp$, s.t.

$\langle\omega_x|w_0\rangle = 1$.

Part of $|w_0\rangle$ does not rotate.

Span programs – algorithm construction

Witnesses of positive and negative instances:

1. Positive witness:
   $|w_x\rangle \in \mathcal{H}(x)$, s.t. $A|w_x\rangle = |\tau\rangle$.
2. Negative witness:
   $|\omega_x\rangle \in \mathcal{H}(x)^\perp \cap \ker(A)^\perp$, s.t. $\langle w_0|\omega_x\rangle = 1$.

Span programs – algorithm construction

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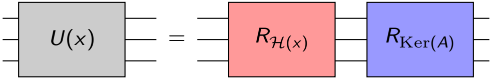
Span program unitary:



Span programs – algorithm construction

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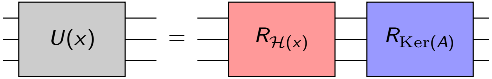

Run phase estimation with initial state $|w_0\rangle / \||w_0\rangle\|$.

Span programs – algorithm construction

Witnesses of positive and negative instances:

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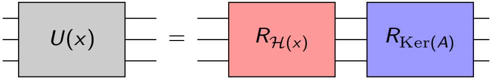
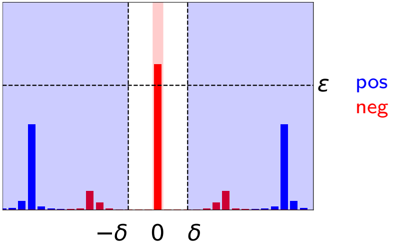


Run phase estimation with initial state $|w_0\rangle / \||w_0\rangle\|$.
Outcome distribution:



with $\delta = \frac{\||w_0\rangle\|}{\||w_x\rangle\|}$ and $\varepsilon = \frac{1}{\||\omega_x\rangle\| \cdot \||w_0\rangle\|}$.

Span programs – algorithm construction

Witnesses of positive and negative instances:

1. Positive witness:
   $|w_x\rangle \in \mathcal{H}(x)$, s.t. $A|w_x\rangle = |\tau\rangle$.
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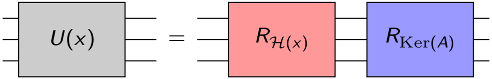
Algorithm compiled from span program:

Span program unitary:



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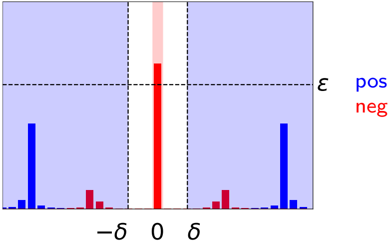

with $\delta = \frac{\||w_0\rangle\|}{\||w_x\rangle\|}$ and $\varepsilon = \frac{1}{\||\omega_x\rangle\| \cdot \||w_0\rangle\|}$.

Span programs – algorithm construction

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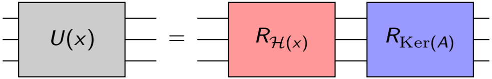
Algorithm compiled from span program:

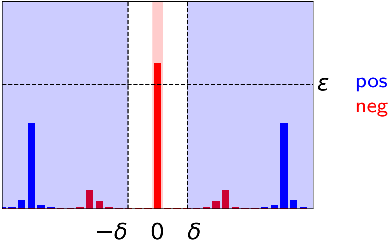
1. Run phase estimation up to precision

$$\delta = \frac{\| |w_0\rangle \|}{\max_{x \in f^{(-1)}(1)} \| |w_x\rangle \|}.$$

Span program unitary:



Run phase estimation with initial state $|w_0\rangle \,/\, \| |w_0\rangle \|$.
Outcome distribution:



with $\delta = \frac{\| |w_0\rangle \|}{\| |w_x\rangle \|}$ and $\varepsilon = \frac{1}{\| |\omega_x\rangle \| \cdot \| |w_0\rangle \|}$.

Span programs – algorithm construction

Witnesses of positive and negative instances:

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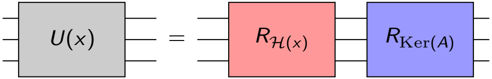
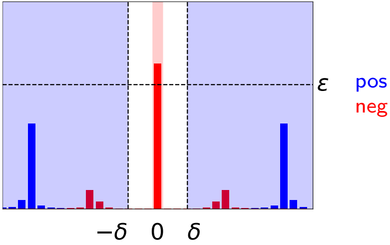
Algorithm compiled from span program:

1. Run phase estimation up to precision
$$\delta = \frac{\||w_0\rangle\|}{\max_{x \in f^{(-1)}(1)} \||w_x\rangle\|}.$$
2. Run amplitude estimation on top of that up to precision
$$\varepsilon = \frac{1}{\||w_0\rangle\| \cdot \max_{x \in f^{(-1)}(0)} \||\omega_x\rangle\|}.$$

Span program unitary:



Run phase estimation with initial state $|w_0\rangle / \||w_0\rangle\|$.
Outcome distribution:



with $\delta = \frac{\||w_0\rangle\|}{\||w_x\rangle\|}$ and $\varepsilon = \frac{1}{\||\omega_x\rangle\| \cdot \||w_0\rangle\|}$.

Span programs – algorithm construction

Witnesses of positive and negative instances:

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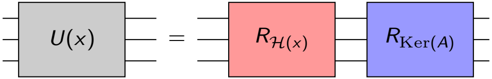
Algorithm compiled from span program:

1. Run phase estimation up to precision
$$\delta = \frac{\||w_0\rangle\|}{\max_{x\in f^{(-1)}(1)} \||w_x\rangle\|}.$$
2. Run amplitude estimation on top of that up to precision
$$\varepsilon = \frac{1}{\||w_0\rangle\| \cdot \max_{x\in f^{(-1)}(0)} \||\omega_x\rangle\|}.$$

No. calls to $U(x)$ is
$$\mathcal{O}\left(\frac{1}{\delta\varepsilon}\right) = \mathcal{O}\left(\max_{x\in f^{(-1)}(1)} \||w_x\rangle\| \cdot \max_{x\in f^{(-1)}(0)} \||\omega_x\rangle\|\right).$$

Span program unitary:



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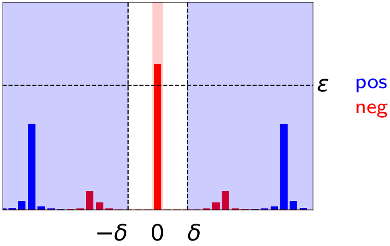


with $\delta = \frac{\||w_0\rangle\|}{\||w_x\rangle\|}$ and $\varepsilon = \frac{1}{\||\omega_x\rangle\|\cdot\||w_0\rangle\|}$.

Span programs – algorithm analysis

Shorthand notation:

$$W_+ = \max_{x \in f^{(-1)}(1)} \left\| |w_x\rangle \right\|^2 \qquad \text{and} \qquad W_- = \max_{x \in f^{(-1)}(0)} \left\| |\omega_x\rangle \right\|^2 .$$

Span programs – algorithm analysis

Shorthand notation:

$$W_+ = \max_{x \in f^{(-1)}(1)} \| |w_x\rangle \|^2 \qquad \text{and} \qquad W_- = \max_{x \in f^{(-1)}(0)} \| |\omega_x\rangle \|^2 .$$

Implementation cost of the algorithm compiled from a span program:

Type	Cost
No. calls to $R_{\ker(A)}$	$\mathcal{O}(\sqrt{W_+ W_-})$
No. calls to $R_{\mathcal{H}(x)}$	$\mathcal{O}(\sqrt{W_+ W_-})$
No. calls to $C_{\|w_0\rangle}$	$\mathcal{O}(\sqrt{W_+ W_-})$
No. calls to $R_{\|0\rangle}$	$\mathcal{O}(\sqrt{W_+ W_-})$
No. extra gates	$\mathcal{O}(\text{polylog}\, W_+ W_-)$
No. extra qubits	$\mathcal{O}(\text{polylog}\, W_+ W_-)$

Span programs compiled from algorithms (I)

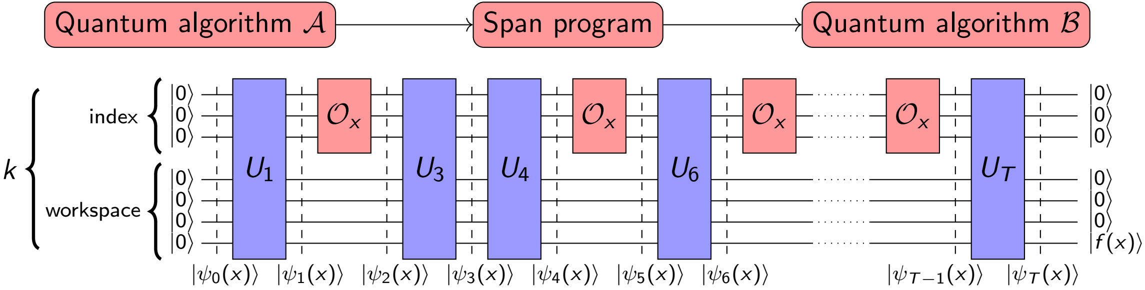
Span programs compiled from algorithms (I)

Span program $\longrightarrow$ Quantum algorithm $\mathcal{B}$

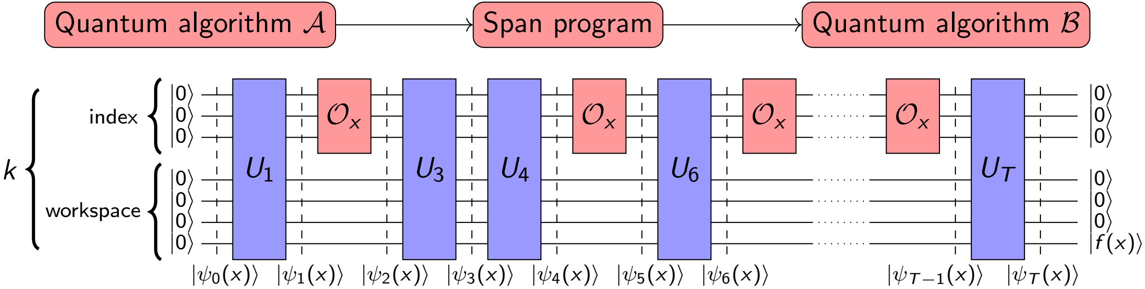
Span programs compiled from algorithms (I)

$$\text{Quantum algorithm } \mathcal{A} \longrightarrow \text{Span program} \longrightarrow \text{Quantum algorithm } \mathcal{B}$$

Span programs compiled from algorithms (I)

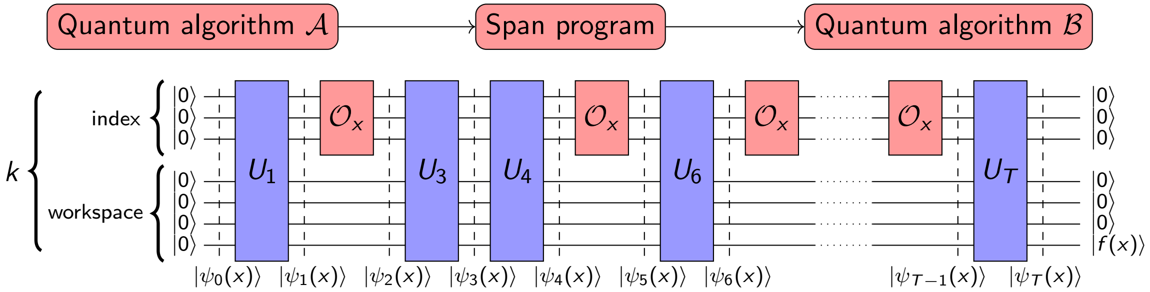


Span programs compiled from algorithms (I)



1. $|\psi_0(x)\rangle = |00\cdots 00\rangle$.
2. $|\psi_T(x)\rangle = |00\cdots 0f(x)\rangle$.

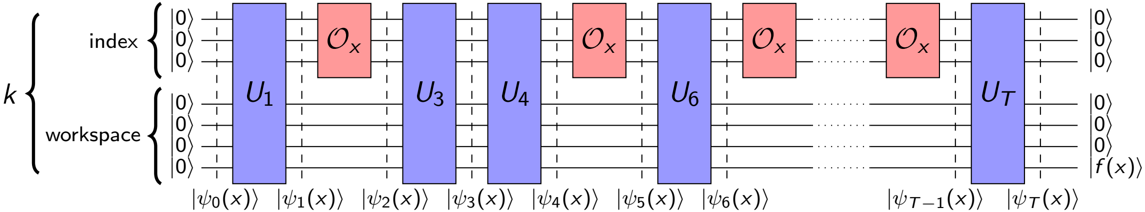
Span programs compiled from algorithms (I)



1. $|\psi_0(x)\rangle = |00\cdots00\rangle$.
2. $|\psi_T(x)\rangle = |00\cdots0f(x)\rangle$.
3. Every U_j can only consist of $\mathcal{O}(\text{polylog}(T))$ elementary gates.

Span programs compiled from algorithms (I)

Quantum algorithm $\mathcal{A}$ $\longrightarrow$ Span program $\longrightarrow$ Quantum algorithm $\mathcal{B}$



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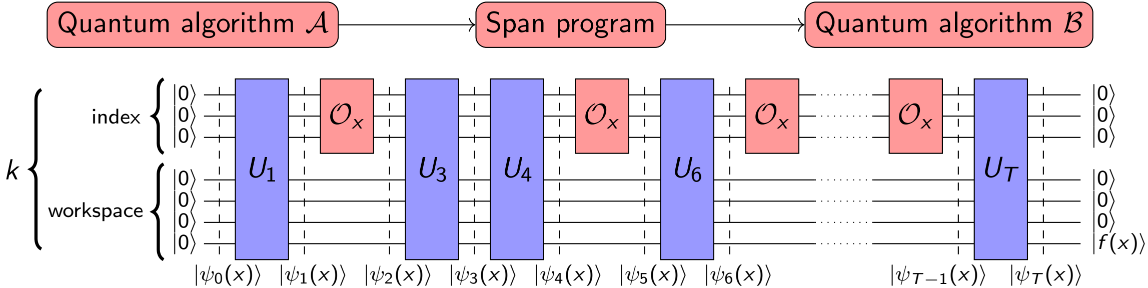
1. S: Query complexity

Span programs compiled from algorithms (I)



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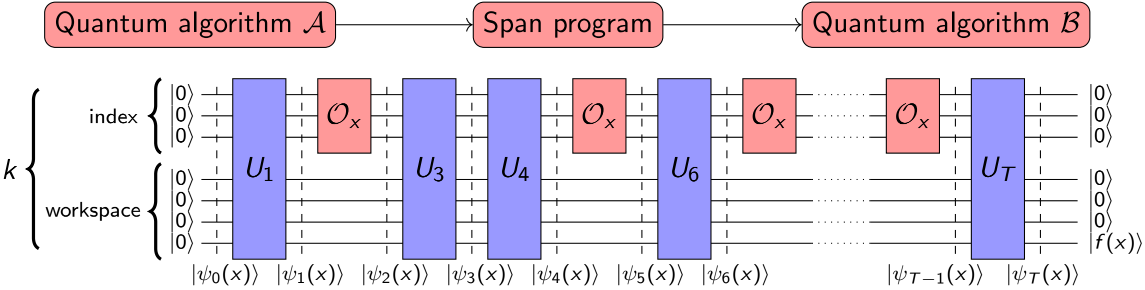
1. S: Query complexity
2. T: No. time steps

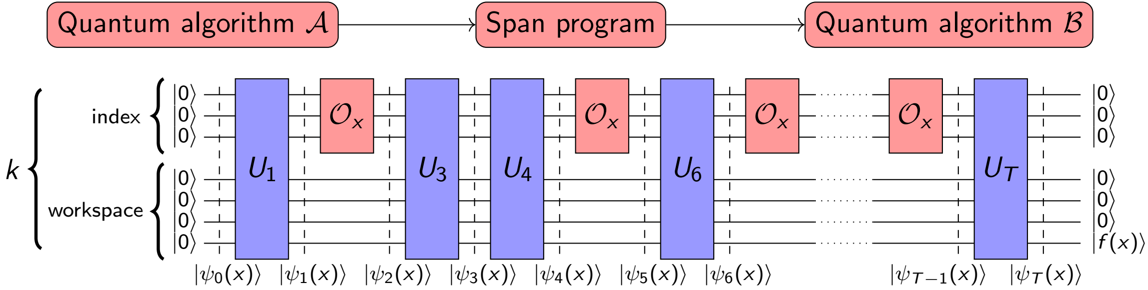
Span programs compiled from algorithms (I)



1. $|\psi_0(x)\rangle = |00\cdots 00\rangle$.
2. $|\psi_T(x)\rangle = |00\cdots 0f(x)\rangle$.
3. Every U_j can only consist of $\mathcal{O}(\text{polylog}(T))$ elementary gates.

1. S: Query complexity
2. T: No. time steps
3. k: No. qubits

Span programs compiled from algorithms (I)



1. $|\psi_0(x)\rangle = |00 \cdots 00\rangle$.
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3. Every U_j can only consist of $\mathcal{O}(\text{polylog}(T))$ elementary gates.

1. S: Query complexity
2. T: No. time steps
3. k: No. qubits
4. ε: Error probability

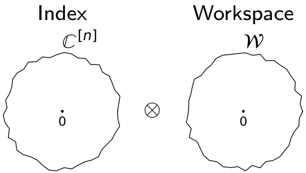
Span programs compiled from algorithms (II)

Span programs compiled from algorithms (II)

Hilbert space $\mathcal{H}$ & target space $\mathcal{V}$:

Span programs compiled from algorithms (II)

Hilbert space $\mathcal{H}$ & target space $\mathcal{V}$:



Span programs compiled from algorithms (II)

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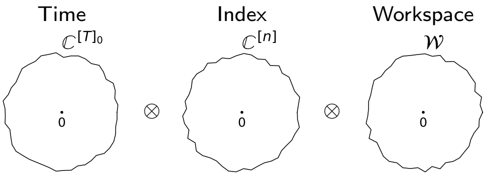

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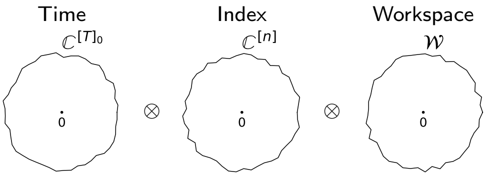

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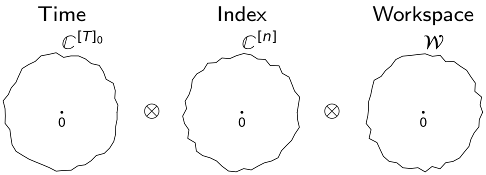

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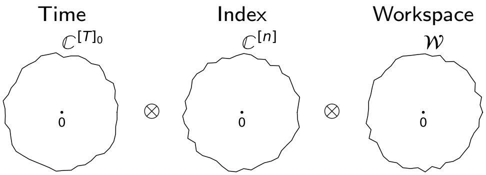

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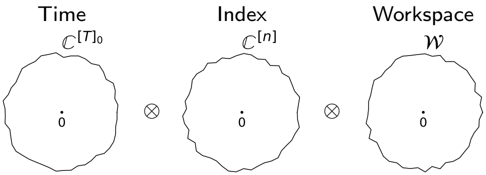

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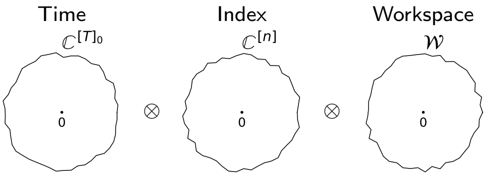

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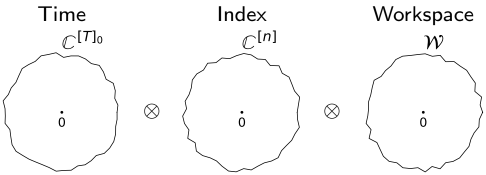

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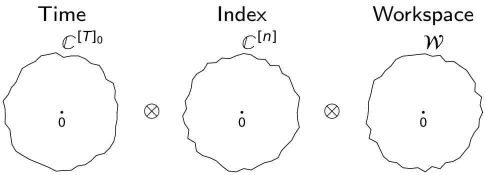

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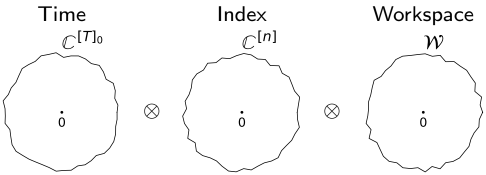

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Analysis of the implementation of the subroutines:

Subroutine	Queries to $\mathcal{O}_x$	Queries to $\mathcal{O}_S$	Queries to $\mathcal{O}_{\mathcal{A}}$	No. extra gates	No. extra qubits	Implementation error
$R_{\ker(A)}$	0	$\mathcal{O}(T/S)$	$\mathcal{O}(T/S)$	$\mathcal{O}(T/S\,\mathrm{polylog}(T))$	$\mathcal{O}(\mathrm{polylog}(T))$	0
$R_{\mathcal{H}(x)}$	$\mathcal{O}(1)$	$\mathcal{O}(1)$	0	$\mathcal{O}(\mathrm{polylog}(T))$	$\mathcal{O}(1)$	0
$C_{\lvert w_0\rangle}$	0	$\mathcal{O}(T/S)$	$\mathcal{O}(T/S)$	$\mathcal{O}(T/S\,\mathrm{polylog}(T))$	$\mathcal{O}(\mathrm{polylog}(T))$	$\mathcal{O}(\sqrt{\varepsilon})$
$R_{\lvert 0\rangle}$	0	0	0	$\mathcal{O}(\log(T))$	$\mathcal{O}(\log(T))$	$\mathcal{O}(\sqrt{\varepsilon})$
Total	$\mathcal{O}(S)$	$\mathcal{O}(T)$	$\mathcal{O}(T)$	$\mathcal{O}(T\,\mathrm{polylog}(T))$	$\mathcal{O}(\mathrm{polylog}(T))$	$\mathcal{O}(S\sqrt{\varepsilon})$
With error red.	$\mathcal{O}(S\log(S))$	$\mathcal{O}(T\log(S))$	$\mathcal{O}(T\log(S))$	$\mathcal{O}(T\,\mathrm{polylog}(T))$	$\mathcal{O}(\mathrm{polylog}(T)+k^{o(1)})$	$\mathcal{O}(\sqrt{\varepsilon})$

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Efficient uniform access: implementation of $\mathcal{O}_{\mathcal{A}}$ and $\mathcal{O}_{\mathcal{S}}$ only takes $\mathcal{O}(\mathrm{polylog}(T))$ gates.

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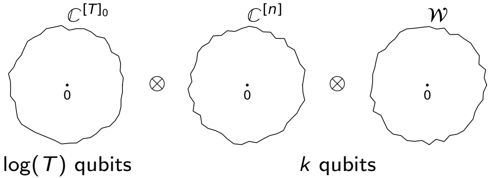

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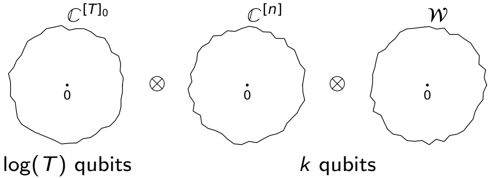

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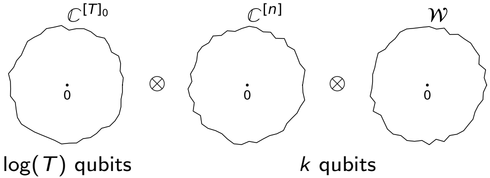

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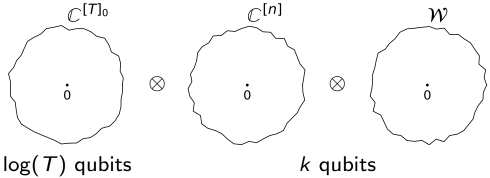

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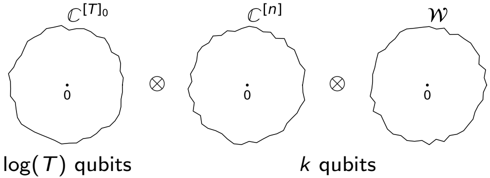

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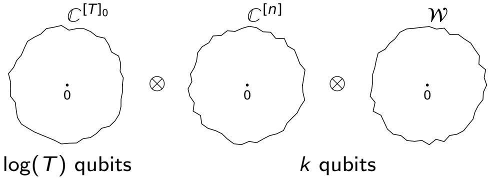

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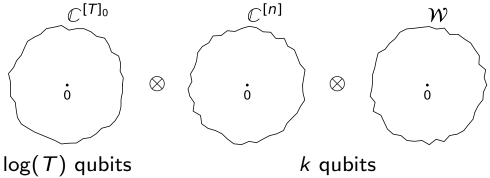
4. Technique of independent interest.



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Now, we define the OR: $f(x^{(1)},\dots,x^{(n)}) = f_1(x^{(1)}) \vee f_2(x^{(2)}) \vee \cdots \vee f_n(x^{(n)})$.
Variable-time search: how quickly can we compute f?

Method	No. queries to $\mathcal{O}_x$	No. queries to $\mathcal{O}_{\mathcal{A}}$ & $\mathcal{O}_{\mathcal{S}}$	No. extra gates
Naive approach	$\sum_{j=1}^n S_j$	$\sum_{j=1}^n T_j$	$\widetilde{\mathcal{O}}(\sum_{j=1}^n T_j)$
Ambainis '06 (I)	$\mathcal{O}\left(\sqrt{\sum_{j=1}^n S_j^2}\right)$	X	?
Ambainis '06 (II)	$\mathcal{O}\left(\sqrt{\sum_{j=1}^n T_j^2}\right)$	$\mathcal{O}\left(\sqrt{\sum_{j=1}^n T_j^2}\right)$	?

Composition of span programs

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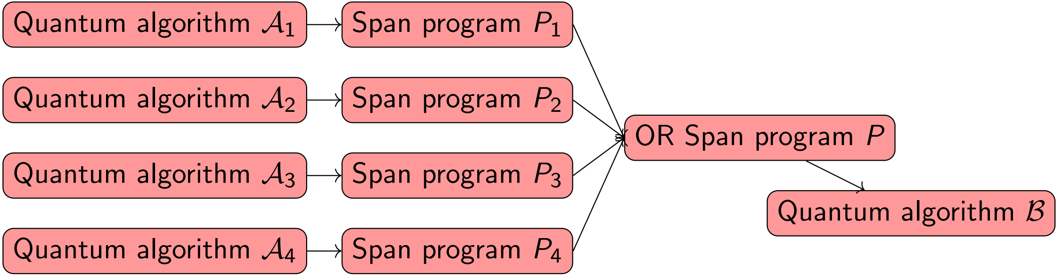

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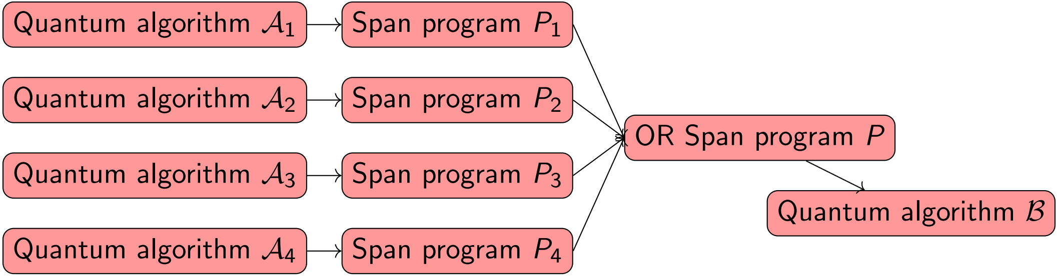

The composition of span programs for *OR* is known (Reichardt, '09).

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Careful construction of the subroutines $R_{\ker(A)}$, $R_{\mathcal{H}(x)}$, $C_{|w_0\rangle}$ and $R_{|0\rangle}$ yields:

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Naive approach	$\sum_{j=1}^n S_j$	$\sum_{j=1}^n T_j$	$\widetilde{\mathcal{O}}(\sum_{j=1}^n T_j)$	$\mathcal{O}\left(\sum_{j=1}^n \varepsilon_j\right)$
Ambainis '06 (I)	$\mathcal{O}\left(\sqrt{\sum_{j=1}^n S_j^2}\right)$	X	?	-
Ambainis '06 (II)	$\mathcal{O}\left(\sqrt{\sum_{j=1}^n T_j^2}\right)$	$\mathcal{O}\left(\sqrt{\sum_{j=1}^n T_j^2}\right)$	?	-
Our result	$\mathcal{O}\left(\sqrt{\sum_{j=1}^n S_j^2}\right)$	$\mathcal{O}\left(\sqrt{\sum_{j=1}^n T_j^2}\right)$	$\widetilde{\mathcal{O}}\left(\sqrt{\sum_{j=1}^n T_j^2}\right)$	$\mathcal{O}\left(\sum_{j=1}^n S_j^2 \sum_{j=1}^n \varepsilon_j\right)$
With error red.	$\mathcal{O}\left(\sqrt{\sum_{j=1}^n S_j^2} \cdot \log\left(n \sum_{j=1}^n S_j^2\right)\right)$	$\mathcal{O}\left(\sqrt{\sum_{j=1}^n T_j^2} \cdot \log\left(n \sum_{j=1}^n S_j^2\right)\right)$	$\widetilde{\mathcal{O}}\left(\sqrt{\sum_{j=1}^n T_j^2}\right)$	$\mathcal{O}(\varepsilon_{\max})$

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Thanks for your attention!