Volume Estimation

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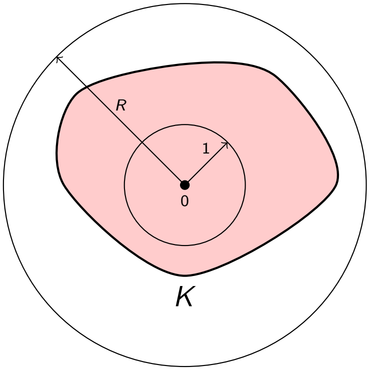

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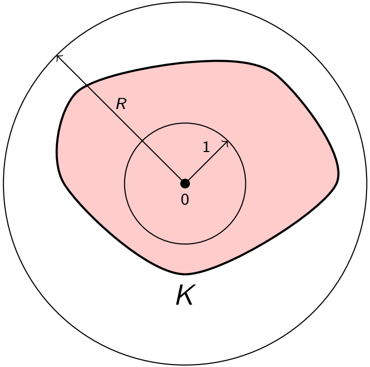

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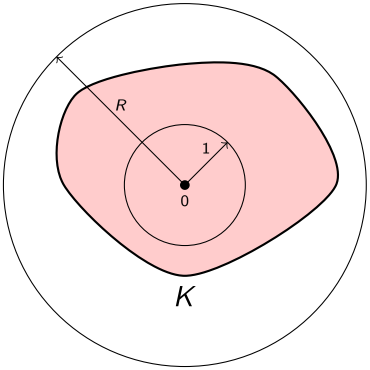

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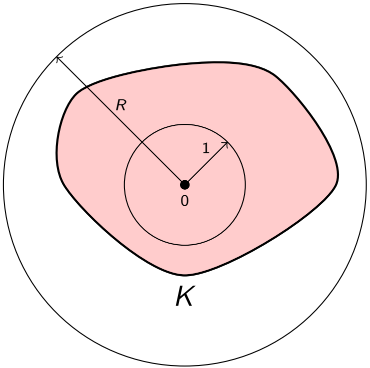

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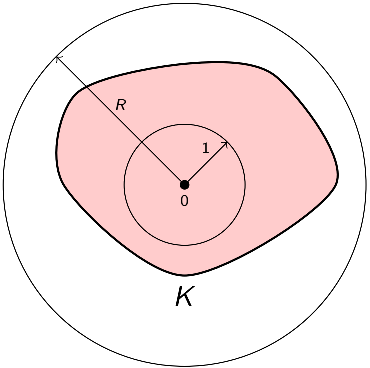
   $O : \mathbb{R}^d \to \{0,1\}, \quad O(x) = [x \in K]$.



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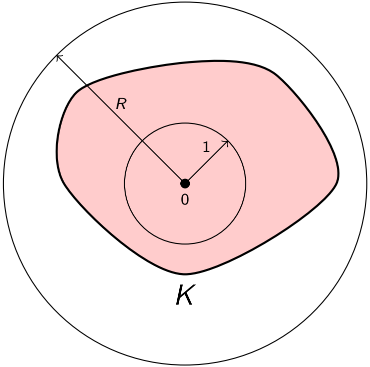
5. *Computational models:*
   1. Deterministic
   2. Randomized (success prob. $\geq 2/3$)
   3. Quantum ($O : |x\rangle |0\rangle \mapsto |x\rangle |x \in K\rangle$)



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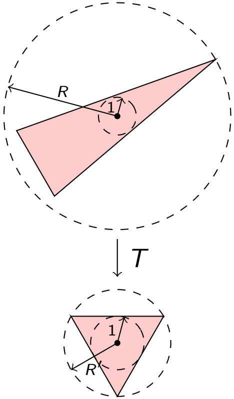

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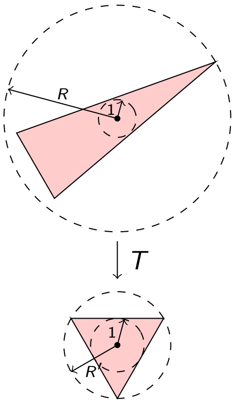

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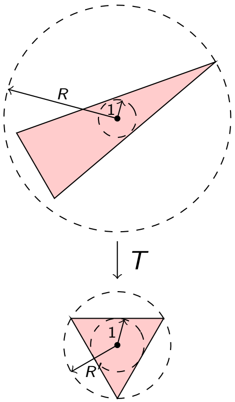

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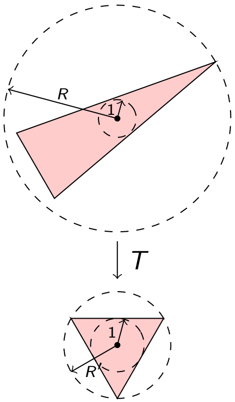

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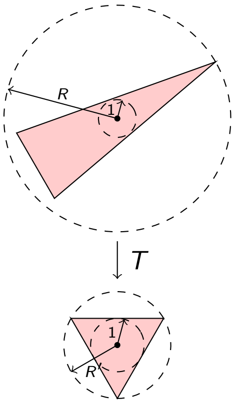

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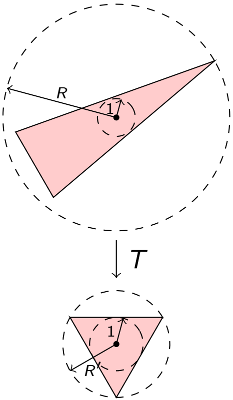

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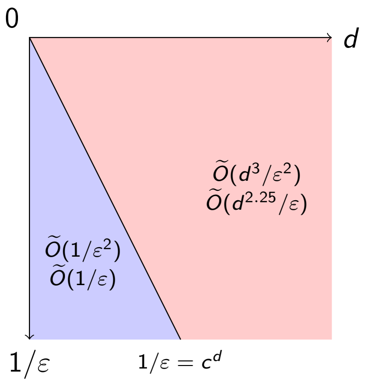

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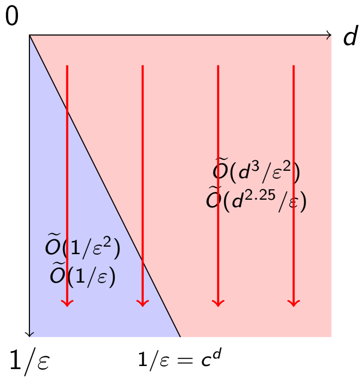

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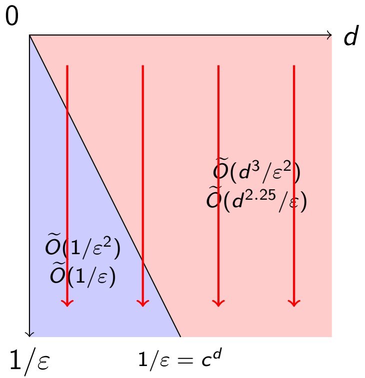
4. *State of the art:*
   1. Randomized: $O(1/\varepsilon^2)$.
   2. Quantum: $O(1/\varepsilon)$.
   3. No lower bounds better than $\Omega(1)$.



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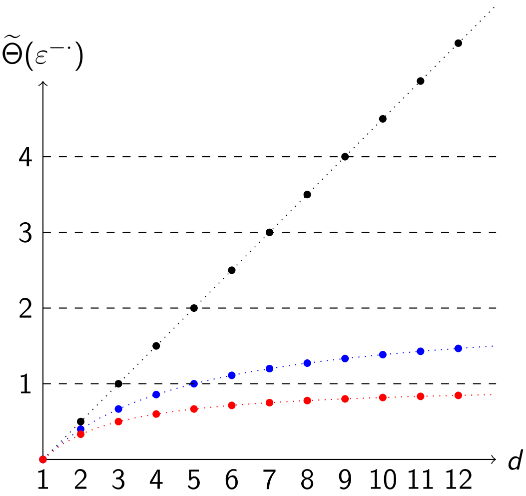

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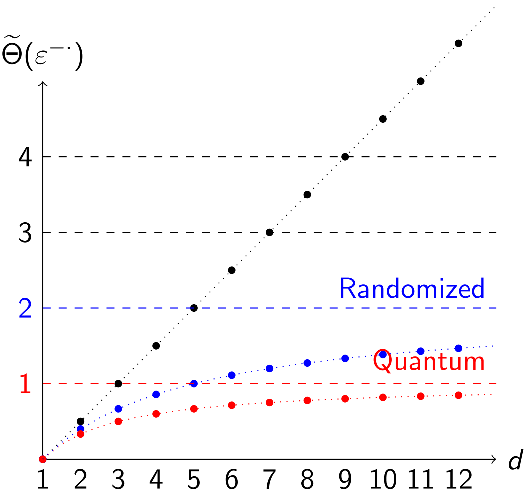
3. *Behavior of the exponent:*
   - $\frac{d-1}{2} \to \infty$.
   - $\frac{2(d-1)}{d+3} = 2 - O(\frac{1}{d}) \to 2$.
   - $\frac{d-1}{d+1} = 1 - O(\frac{1}{d}) \to 1$.



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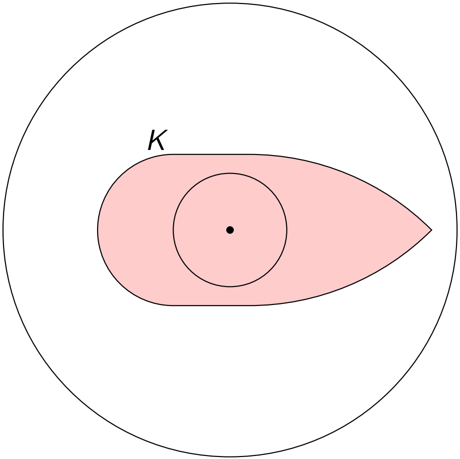

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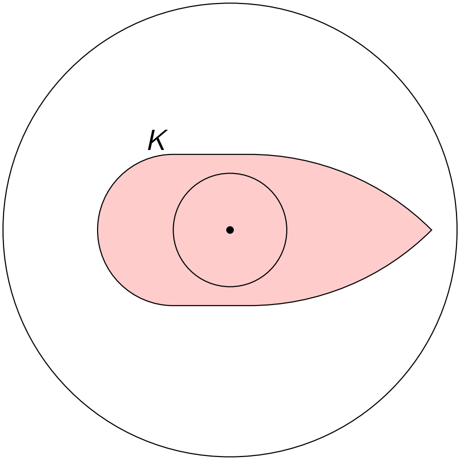

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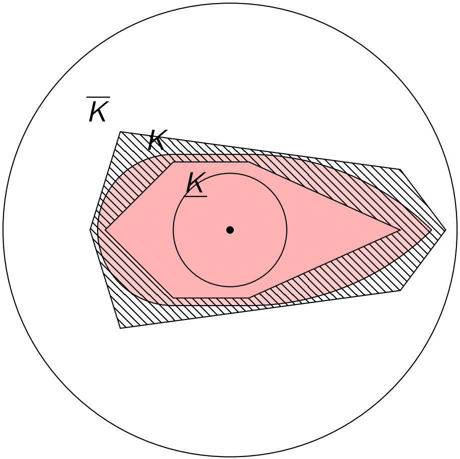

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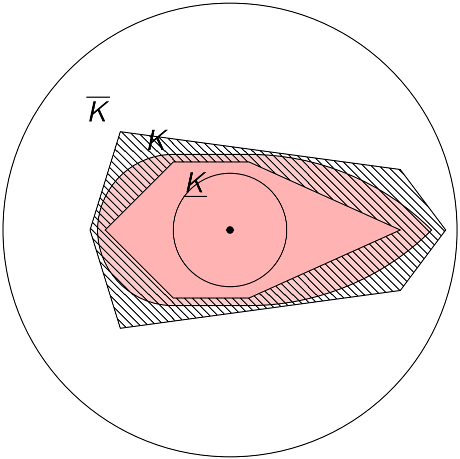

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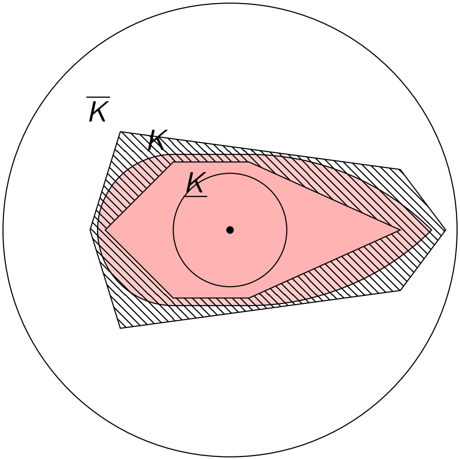

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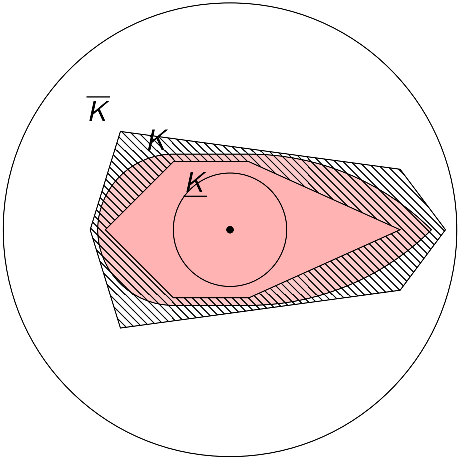

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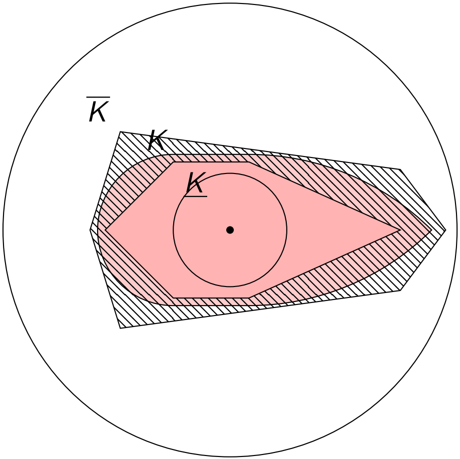

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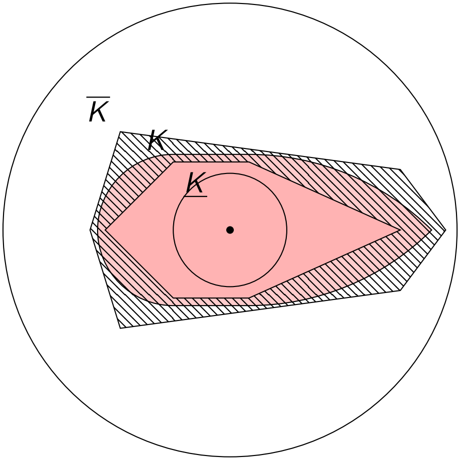

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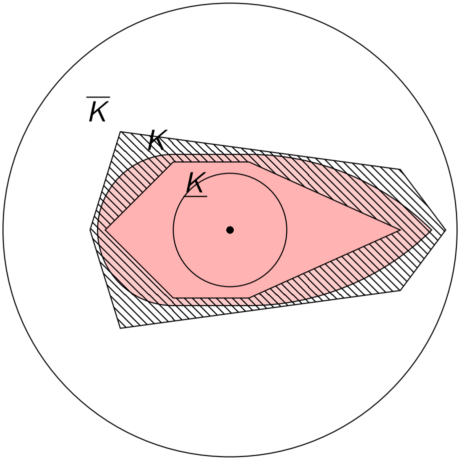
4. *Balance:* Optimize δ.

   $\Rightarrow$ All complexities follow.



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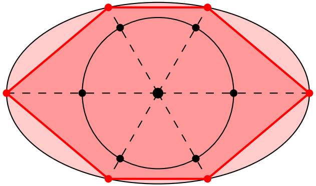
Failed attempt I:

1. Take $v_1, \dots, v_n$ an η-net on $\partial B(0,1)$.
2. Find the boundary points $r_j v_j \in \partial K$ with binary search.
3. $\underline{K} = \mathrm{conv}(r_j v_j)$.
4. Hard to bound volume difference.



Techniques II – Convex set reconstruction attempts

Goal: Find $\underline{K} \subseteq K \subseteq \overline{K}$ s.t. $\mathrm{Vol}(\overline{K} \setminus \underline{K}) \leq \delta$.

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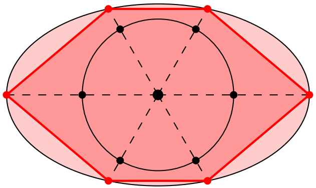
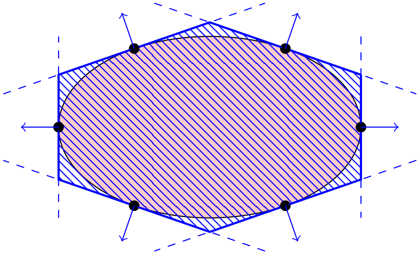


Failed attempt II:

1. Take $v_1, \dots, v_n$ an η-net on $\partial B(0,1)$.
2. Optimize over $x \mapsto v_j^T x$ over K.
3. $\overline{K}$ is the intersection of the halfspaces.
4. Hard to bound volume difference.



Techniques II – Convex set reconstruction attempts

Goal: Find $\underline{K} \subseteq K \subseteq \overline{K}$ s.t. $\mathrm{Vol}(\overline{K} \setminus \underline{K}) \leq \delta$.

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1. Take $v_1, \dots, v_n$ an η-net on $\partial B(0,1)$.
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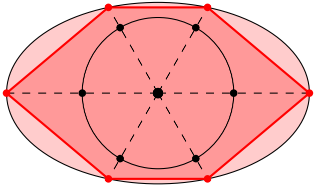

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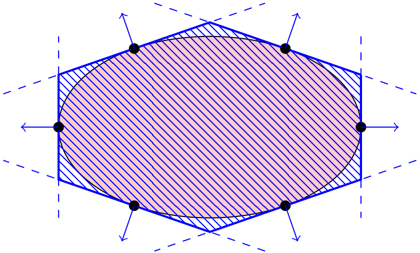

Successful attempt: "average" of the two.

Techniques III – Convex set reconstruction [Dud74]

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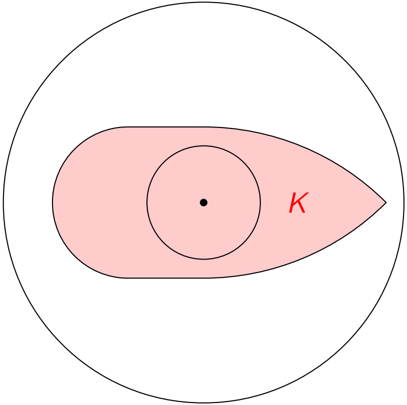

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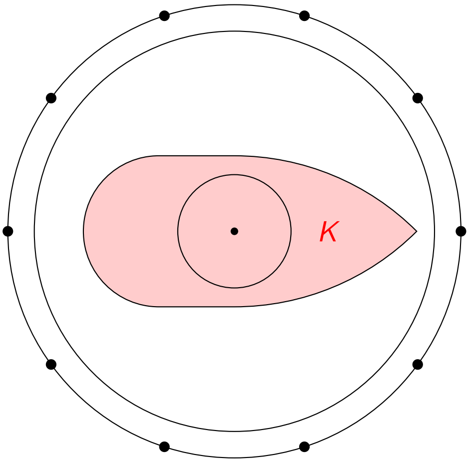
2. *Procedure sketch:*
    1. Take an η-net on $\partial B(0, R+1)$.



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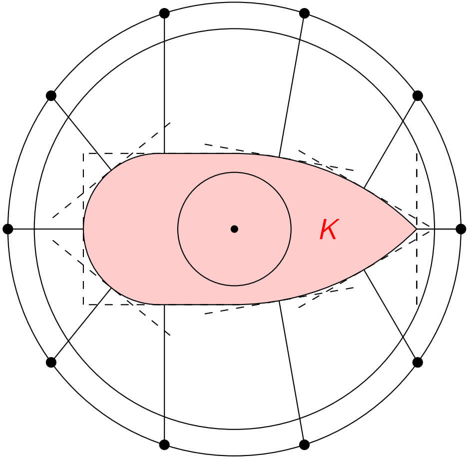

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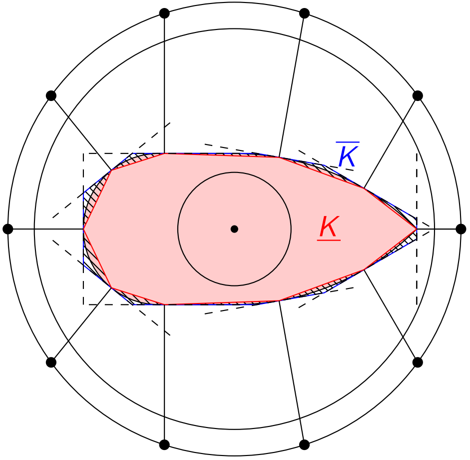

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3. *Approximation claims:* (next slide)
   1. $K \subseteq \underline{K} + B(0, O(\eta^2))$
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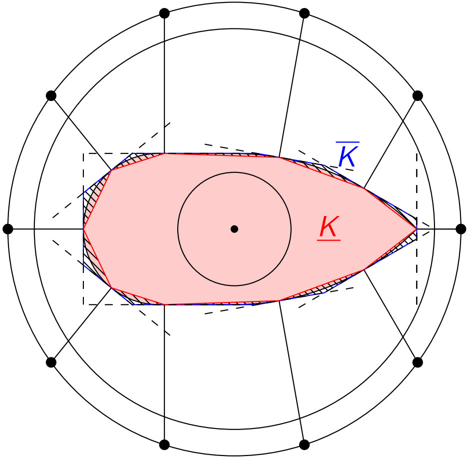

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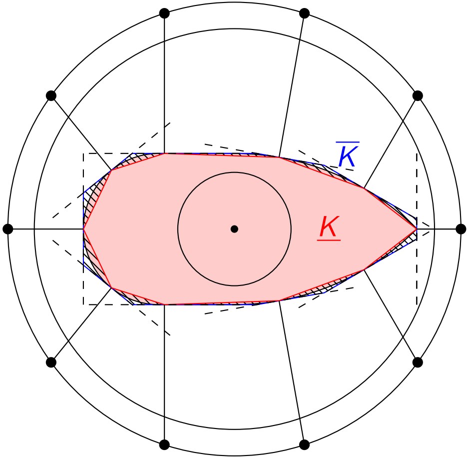

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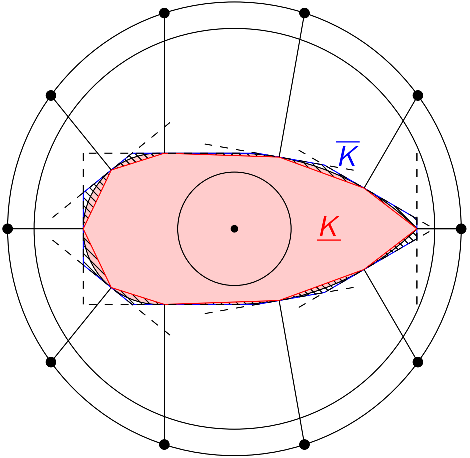

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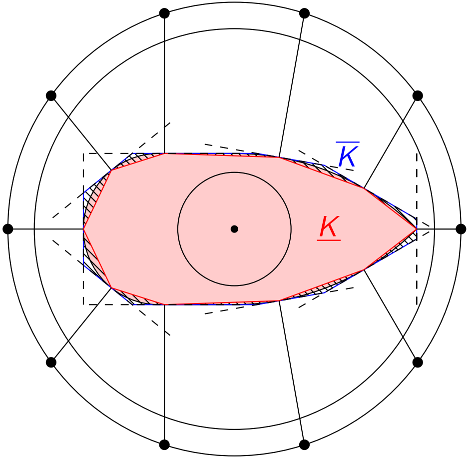

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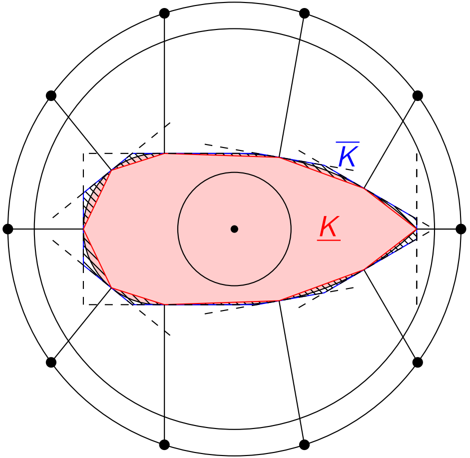
$\Rightarrow C_\delta = \widetilde{O}(\delta^{-\frac{d-1}{2}})$.



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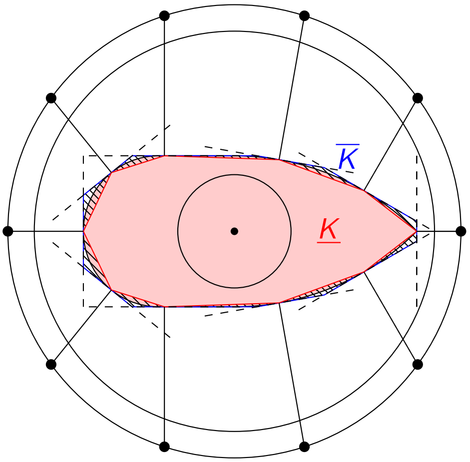
   $\Rightarrow C_\delta = \widetilde{O}(\delta^{-\frac{d-1}{2}})$.
5. *Remark:* Approximation errors.



Techniques IV – Approximation proof

1. *Goal:* $\overline{K} \subseteq K + B(0, O(\eta^2))$.

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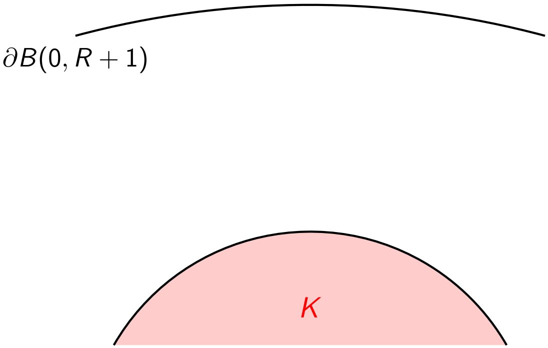

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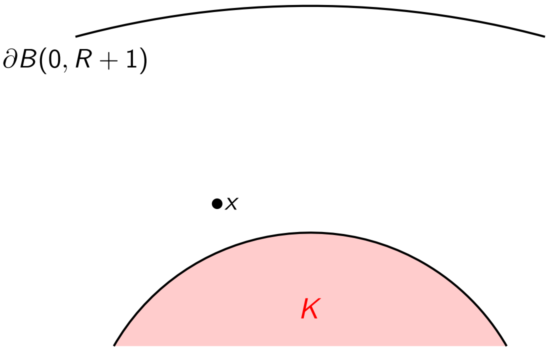

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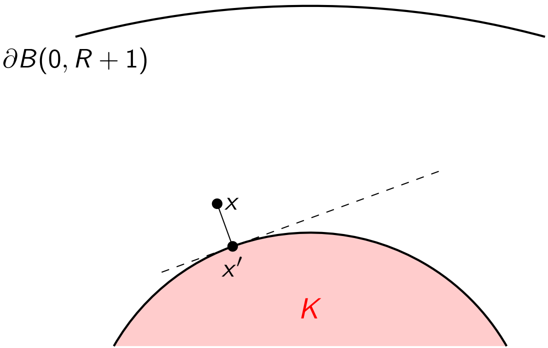

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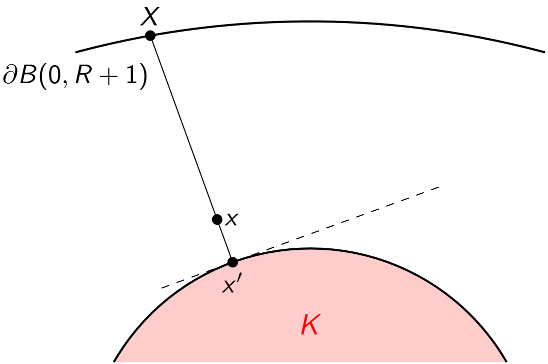

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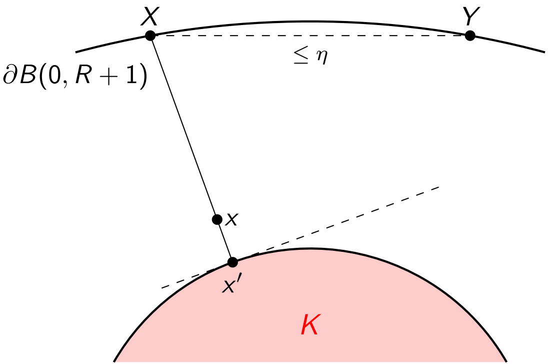

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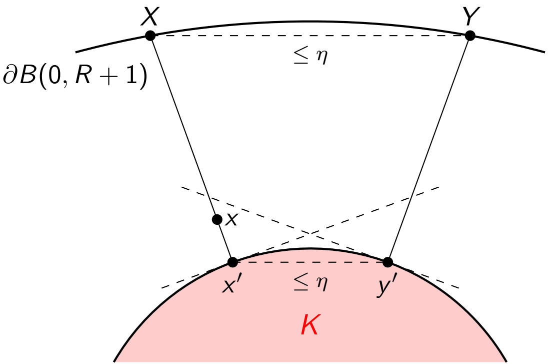

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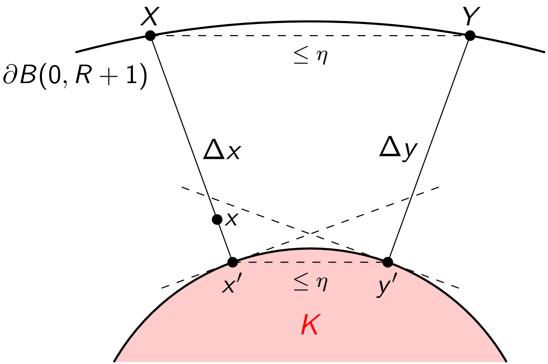
   6. Let $\Delta x = X - x'$, $\Delta y = Y - y'$.



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3. *Observations:*
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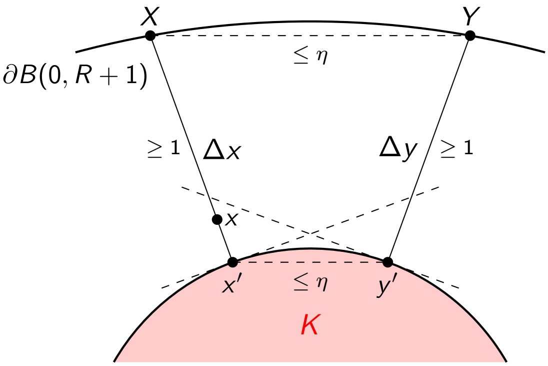

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3. *Observations:*
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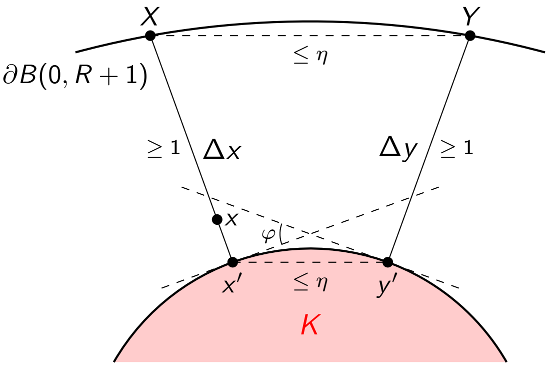
   2. $\Rightarrow \varphi = \angle(\Delta x, \Delta y) = O(\eta)$.



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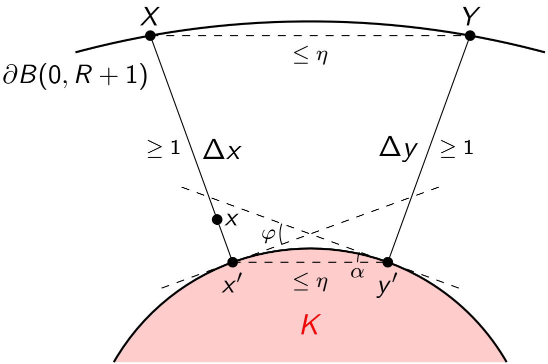
   3. $\alpha \leq \pi - (\pi - \varphi) = \varphi$.



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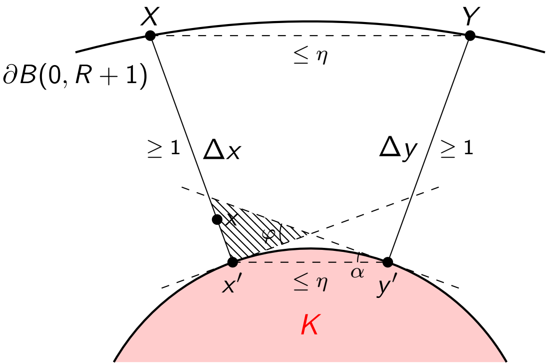

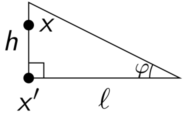

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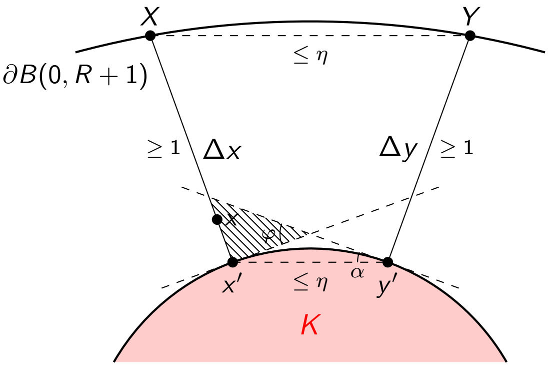

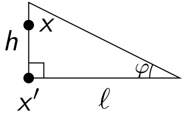

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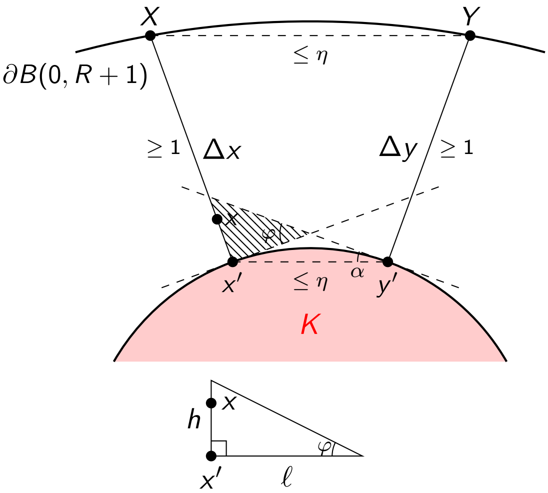

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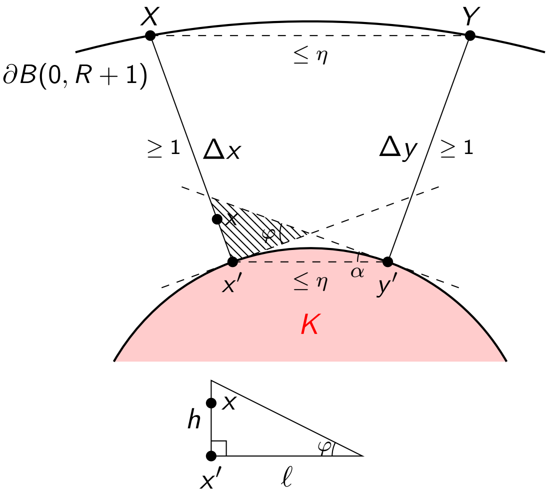
   4. Find Y from the η-net s.t. $\|Y - X\| \leq \eta$.
   5. Let y' be the projection of Y onto K.
   6. Let $\Delta x = X - x'$, $\Delta y = Y - y'$.
3. *Observations:*
   1. $\|\Delta x - \Delta y\| \leq 2\eta$, $\|\Delta x\| \geq 1$, $\|\Delta y\| \geq 1$.
   2. $\Rightarrow \varphi = \angle(\Delta x, \Delta y) = O(\eta)$. [Attempt II]
   3. $\alpha \leq \pi - (\pi - \varphi) = \varphi$.
   4. $\Rightarrow \ell = \|x' - y'\| \cdot \frac{\sin(\alpha)}{\sin(\varphi)} = O(\eta)$. [Attempt I]
   5. $\Rightarrow h = \ell \tan(\varphi) = O(\eta^2)$. □



Techniques V – Lower bounds

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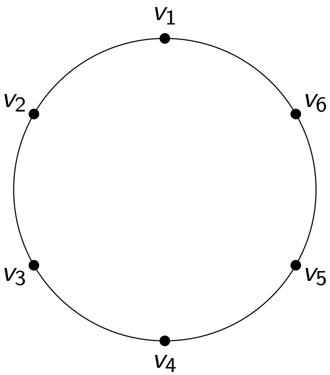

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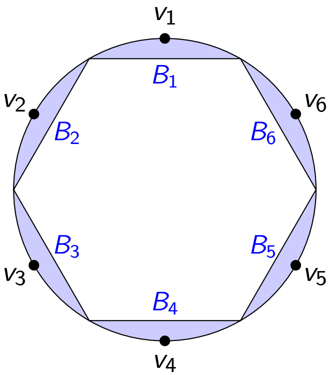

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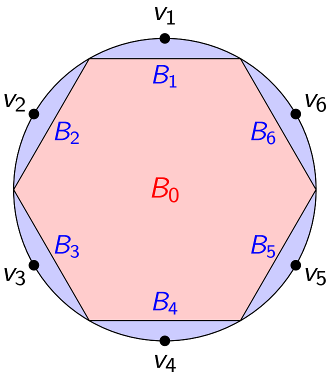

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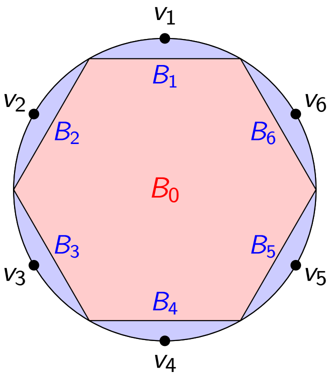

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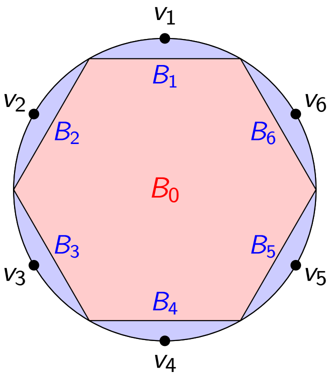

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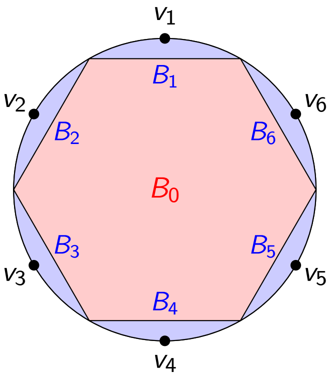

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4. *Balance:* Optimize $\eta \Rightarrow$ All bounds follow.

Summary & Outlook

1. *Our results:* (d fixed, $\varepsilon \downarrow 0$)

Model	Reconstruction	Volume est.
Deterministic	$\widetilde{\Theta}(\varepsilon^{-\frac{d-1}{2}})$	$\widetilde{\Theta}(\varepsilon^{-\frac{d-1}{2}})$
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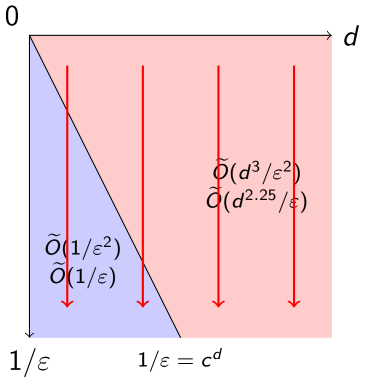

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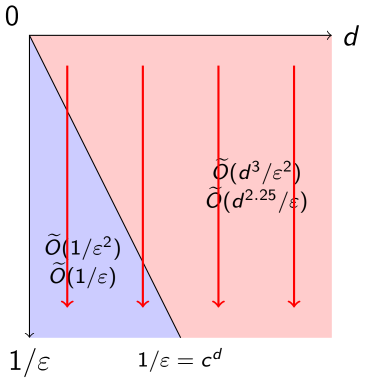

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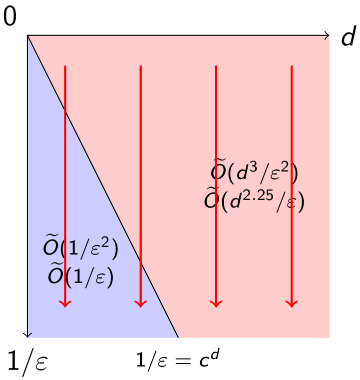

Thanks for your attention!

cornelissen@irif.fr

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