Quantum Sabotage Complexity

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Complexity theory

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   They quantify complexity.

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   Examples:
   1. $\mathrm{D}(f)$: Deterministic q.c.
   2. $\mathrm{R}(f)$: Randomized q.c.
   3. $\mathrm{Q}(f)$: Quantum q.c.

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4. *Follow-up question:*
   How do these measures relate?
   1. *Hasse diagram*

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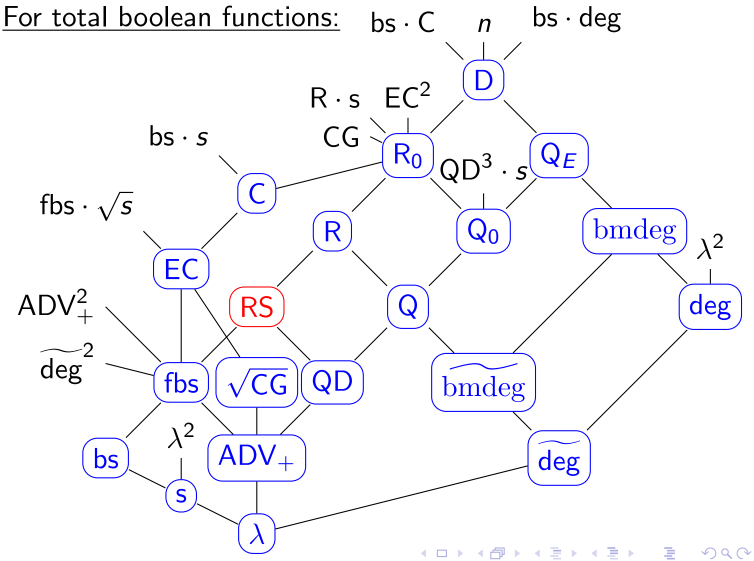
<u>For total boolean functions:</u>



Sabotage complexity [BK16]

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j	1	2	3	$\cdots$	n
x	0	0	1	$\cdots$	1
y	0	1	0	$\cdots$	1

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$$
\begin{array}{r|ccccc}
j & 1 & 2 & 3 & \cdots & n \\
\hline
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\\
\downarrow & & & & & \\
z_{x,y,*} & 0 & * & * & \cdots & 1 \\
z_{x,y,\dagger} & 0 & \dagger & \dagger & \cdots & 1
\end{array}
$$

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Boolean function: $f : \{0,1\}^n \to \{0,1\}$.

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$$(z_{x,y,*/\dagger})_j = \begin{cases} x_j, & \text{if } x_j = y_j, \\ */\dagger, & \text{if } x_j \neq y_j. \end{cases}$$

j	1	2	3	$\cdots$	n
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3. $f_{\text{sab}} : z_{x,y,*/\dagger} \mapsto */\dagger$.

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3. $f_{\mathrm{sab}} : z_{x,y,*/\dagger} \mapsto */\dagger$.
4. $\mathrm{DS}(f) := \mathrm{D}(f_{\mathrm{sab}})$, $\mathrm{RS}(f) := \mathrm{R}_0(f_{\mathrm{sab}})$.

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Their observations:

1. $\mathrm{DS}(f) = \mathrm{D}(f)$. [BK16; Theorem 33]
2. $\mathrm{RS}(f) = O(\mathrm{R}(f))$. [BK16; Theorem 12]

j	1	2	3	$\cdots$	n	
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$z_{x,y,*}$	0	$*$	$*$	$\cdots$	1	$\mapsto *$
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Open question:

1. Quantum analog: $\text{QS}(f) = O(\text{Q}(f))$?

Quantum sabotage complexity

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Boolean function: $f : \{0,1\}^n \to \{0,1\}$.

1. Let $x \in f^{-1}(0)$, $y \in f^{-1}(1)$.
2. Let z be a sabotaged input.
3. Let $J_z = \{j : z_j \in \{*, \dagger\}\}$.

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		J_z			
j	1	2	3	$\cdots$	n
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Complexity measure definitions:

	$\mathrm{D}(\cdot)$	$\mathrm{R}(\cdot)$	$\mathrm{Q}(\cdot)$

		J_z			
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Complexity measure definitions:

	$\mathrm{D}(\cdot)$	$\mathrm{R}(\cdot)$	$\mathrm{Q}(\cdot)$
$f_{\mathrm{sab,weak}} : z \mapsto */\dagger$	$\mathrm{DS}_{\mathrm{weak}}$	$\mathrm{RS}_{\mathrm{weak}}$	$\mathrm{QS}_{\mathrm{weak}}$

$$\begin{array}{c|ccccc}
 & & \multicolumn{2}{c}{J_z} & & \\
j & 1 & 2 & 3 & \cdots & n \\
\hline
z & 0 & * & * & \cdots & 1 \quad \mapsto */\dagger
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$f_{\mathrm{sab,weak}} : z \mapsto */\dagger$	$\mathrm{DS}_{\mathrm{weak}}$	$\mathrm{RS}_{\mathrm{weak}}$	$\mathrm{QS}_{\mathrm{weak}}$
$f^{\mathrm{ind}}_{\mathrm{sab,weak}}(z) \in J_z$	$\mathrm{DS}^{\mathrm{ind}}_{\mathrm{weak}}$	$\mathrm{RS}^{\mathrm{ind}}_{\mathrm{weak}}$	$\mathrm{QS}^{\mathrm{ind}}_{\mathrm{weak}}$

$$\begin{array}{c|ccccc}
 & & \multicolumn{2}{c}{J_z} & & \\
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 & & & \downarrow & & \\
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$f^{\mathrm{ind}}_{\mathrm{sab,weak}}(z) \in J_z$	$\mathrm{DS}^{\mathrm{ind}}_{\mathrm{weak}}$	$\mathrm{RS}^{\mathrm{ind}}_{\mathrm{weak}}$	$\mathrm{QS}^{\mathrm{ind}}_{\mathrm{weak}}$
$f_{\mathrm{sab,str}} : (x,y,z) \mapsto */\dagger$	$\mathrm{DS}_{\mathrm{str}}$	$\mathrm{RS}_{\mathrm{str}}$	$\mathrm{QS}_{\mathrm{str}}$

j	1	2 (J_z)	3 (J_z)	$\cdots$	n	
x	0	0	1	$\cdots$	1	
y	0	1	0	$\cdots$	1	
z	0	$*$	$*$	$\cdots$	1	$\mapsto */\dagger$

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$f_{\mathrm{sab,str}} : (x,y,z) \mapsto */\dagger$	$\mathrm{DS}_{\mathrm{str}}$	$\mathrm{RS}_{\mathrm{str}}$	$\mathrm{QS}_{\mathrm{str}}$
$f^{\mathrm{ind}}_{\mathrm{sab,str}}(x,y,z) \in J_z$	$\mathrm{DS}^{\mathrm{ind}}_{\mathrm{str}}$	$\mathrm{RS}^{\mathrm{ind}}_{\mathrm{str}}$	$\mathrm{QS}^{\mathrm{ind}}_{\mathrm{str}}$

		J_z			
j	1	2	3	$\cdots$	n
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			$\downarrow$ j		

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Complexity measure definitions:

	$D(\cdot)$	$R(\cdot)$	$Q(\cdot)$
$f_{\text{sab,weak}} : z \mapsto */\dagger$	DS_{weak}	RS_{weak}	QS_{weak}
$f^{\text{ind}}_{\text{sab,weak}}(z) \in J_z$	$\text{DS}^{\text{ind}}_{\text{weak}}$	$\text{RS}^{\text{ind}}_{\text{weak}}$	$\text{QS}^{\text{ind}}_{\text{weak}}$
$f_{\text{sab,str}} : (x,y,z) \mapsto */\dagger$	DS_{str}	RS_{str}	QS_{str}
$f^{\text{ind}}_{\text{sab,str}}(x,y,z) \in J_z$	$\text{DS}^{\text{ind}}_{\text{str}}$	$\text{RS}^{\text{ind}}_{\text{str}}$	$\text{QS}^{\text{ind}}_{\text{str}}$

		J_z			
j	1	2	3	$\cdots$	n
x	0	0	1	$\cdots$	1
y	0	1	0	$\cdots$	1
z	0	$*$	$*$	$\cdots$	1

Observations:

1. $\text{DS} = \Theta(\text{DS}_{\text{weak}}) = \Theta(\text{DS}^{\text{ind}}_{\text{weak}}) = \Theta(\text{DS}_{\text{str}}) = \Theta(\text{DS}^{\text{ind}}_{\text{str}})$.
2. $\text{RS} = \Theta(\text{RS}_{\text{weak}}) = \Theta(\text{RS}^{\text{ind}}_{\text{weak}}) = \Theta(\text{RS}_{\text{str}}) = \Theta(\text{RS}^{\text{ind}}_{\text{str}})$.

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Complexity measure definitions:

	$D(\cdot)$	$R(\cdot)$	$Q(\cdot)$
$f_{\text{sab,weak}} : z \mapsto */\dagger$	DS_{weak}	RS_{weak}	QS_{weak}
$f_{\text{sab,weak}}^{\text{ind}}(z) \in J_z$	$DS_{\text{weak}}^{\text{ind}}$	$RS_{\text{weak}}^{\text{ind}}$	$QS_{\text{weak}}^{\text{ind}}$
$f_{\text{sab,str}} : (x,y,z) \mapsto */\dagger$	DS_{str}	RS_{str}	QS_{str}
$f_{\text{sab,str}}^{\text{ind}}(x,y,z) \in J_z$	$DS_{\text{str}}^{\text{ind}}$	$RS_{\text{str}}^{\text{ind}}$	$QS_{\text{str}}^{\text{ind}}$

		J_z			
j	1	2	3	$\cdots$	n
x	0	0	1	$\cdots$	1
y	0	1	0	$\cdots$	1
z	0	$*$	$*$	$\cdots$	1

Observations:

1. $DS = \Theta(DS_{\text{weak}}) = \Theta(DS_{\text{weak}}^{\text{ind}}) = \Theta(DS_{\text{str}}) = \Theta(DS_{\text{str}}^{\text{ind}})$.
2. $RS = \Theta(RS_{\text{weak}}) = \Theta(RS_{\text{weak}}^{\text{ind}}) = \Theta(RS_{\text{str}}) = \Theta(RS_{\text{str}}^{\text{ind}})$.

Question: How about the quantum versions?

Results

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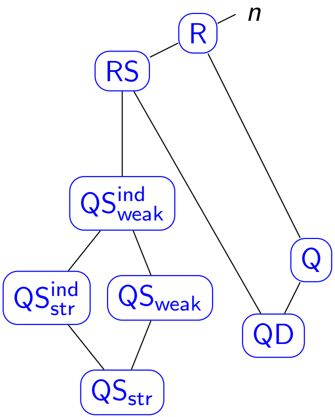
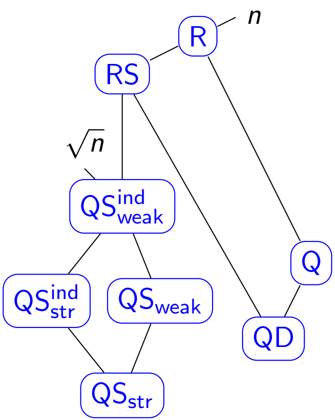
1. *Direct inclusions:*
   1. $\mathrm{QS}_{\mathrm{str}} = O(\mathrm{QS}^{\mathrm{ind}}_{\mathrm{str}}) = O(\mathrm{QS}^{\mathrm{ind}}_{\mathrm{weak}}) = O(\mathrm{RS})$.
   2. $\mathrm{QS}_{\mathrm{str}} = O(\mathrm{QS}_{\mathrm{weak}}) = O(\mathrm{QS}^{\mathrm{ind}}_{\mathrm{weak}})$.



Results

1. *Direct inclusions:*
    1. $\mathrm{QS}_{\mathrm{str}} = O(\mathrm{QS}^{\mathrm{ind}}_{\mathrm{str}}) = O(\mathrm{QS}^{\mathrm{ind}}_{\mathrm{weak}}) = O(\mathrm{RS})$.
    2. $\mathrm{QS}_{\mathrm{str}} = O(\mathrm{QS}_{\mathrm{weak}}) = O(\mathrm{QS}^{\mathrm{ind}}_{\mathrm{weak}})$.
2. *Search upper bound:* $\mathrm{QS}^{\mathrm{ind}}_{\mathrm{weak}} = O(\sqrt{n})$.



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   1. $\mathrm{QS}_{\mathrm{str}} = O(\mathrm{QS}^{\mathrm{ind}}_{\mathrm{str}}) = O(\mathrm{QS}^{\mathrm{ind}}_{\mathrm{weak}}) = O(\mathrm{RS})$.
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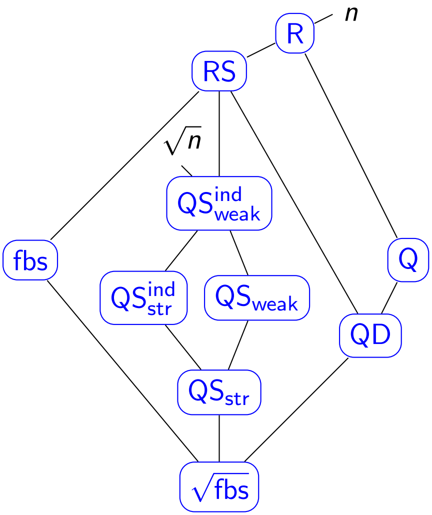
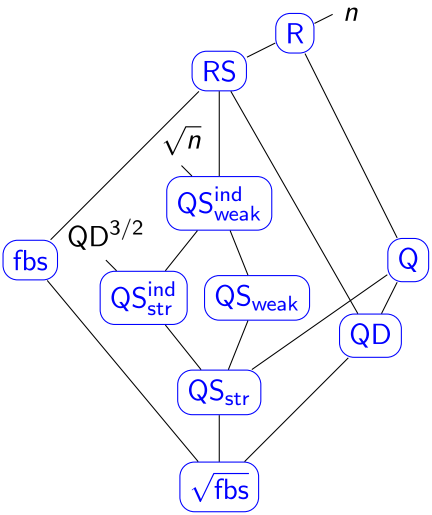
2. *Search upper bound:* $\mathrm{QS}^{\mathrm{ind}}_{\mathrm{weak}} = O(\sqrt{n})$.
3. *Lower bound:* $\mathrm{QS}_{\mathrm{str}} = \Omega(\sqrt{\mathrm{fbs}})$.



Results

1. *Direct inclusions:*
   1. $\mathrm{QS}_{\mathrm{str}} = O(\mathrm{QS}^{\mathrm{ind}}_{\mathrm{str}}) = O(\mathrm{QS}^{\mathrm{ind}}_{\mathrm{weak}}) = O(\mathrm{RS})$.
   2. $\mathrm{QS}_{\mathrm{str}} = O(\mathrm{QS}_{\mathrm{weak}}) = O(\mathrm{QS}^{\mathrm{ind}}_{\mathrm{weak}})$.
2. *Search upper bound:* $\mathrm{QS}^{\mathrm{ind}}_{\mathrm{weak}} = O(\sqrt{n})$.
3. *Lower bound:* $\mathrm{QS}_{\mathrm{str}} = \Omega(\sqrt{\mathrm{fbs}})$.
4. *Algorithmic relations:*
   1. $\mathrm{QS}_{\mathrm{str}} = O(\mathrm{Q})$.
      Desired property from [BK16].
   2. $\mathrm{QS}^{\mathrm{ind}}_{\mathrm{str}} = O(\mathrm{QD}^{3/2})$.



Results

1. *Direct inclusions:*
   1. $\mathrm{QS_{str}} = O(\mathrm{QS^{ind}_{str}}) = O(\mathrm{QS^{ind}_{weak}}) = O(\mathrm{RS})$.
   2. $\mathrm{QS_{str}} = O(\mathrm{QS_{weak}}) = O(\mathrm{QS^{ind}_{weak}})$.
2. *Search upper bound:* $\mathrm{QS^{ind}_{weak}} = O(\sqrt{n})$.
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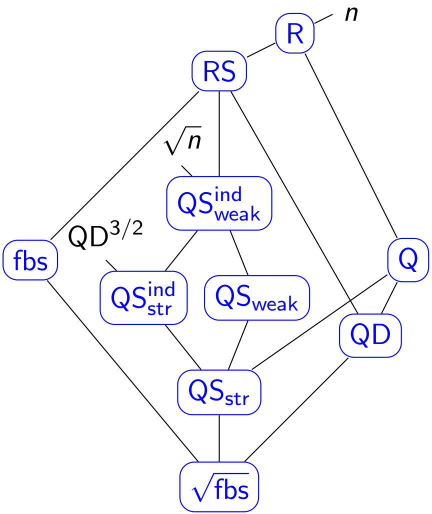
4. *Algorithmic relations:*
   1. $\mathrm{QS_{str}} = O(\mathrm{Q})$.
      Desired property from [BK16].
   2. $\mathrm{QS^{ind}_{str}} = O(\mathrm{QD}^{3/2})$.
5. *Separation:* $\exists f : \mathrm{QS_{str}}(f) = \Omega(\mathrm{fbs}(f))$.



Algorithmic relation I: $\mathrm{QS}_{\mathrm{str}} = O(\mathrm{Q})$

Boolean function: $f : \{0,1\}^n \to \{0,1\}$.

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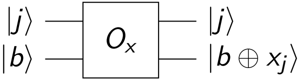
1. *Algorithm for f:* $\mathcal{A}$
   1. makes oracle calls $O_x : j \mapsto x_j$,
   2. for any $x \in \{0,1\}^n$, $\mathcal{A}(x) = f(x)$ whp.



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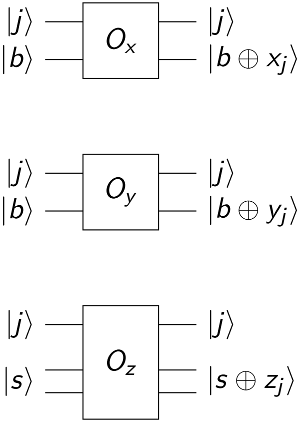
   1. makes oracle calls $O_x : j \mapsto x_j$,
   2. for any $x \in \{0,1\}^n$, $\mathcal{A}(x) = f(x)$ whp.
2. *Strong model:* $f_{\mathrm{sab,str}} : (x, y, z_{x,y,*/\dagger}) \mapsto */\dagger$.



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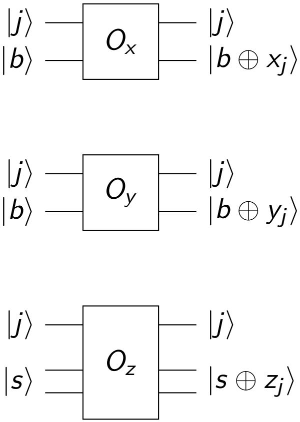
1. *Algorithm for f:* $\mathcal{A}$
   1. makes oracle calls $O_x : j \mapsto x_j$,
   2. for any $x \in \{0,1\}^n$, $\mathcal{A}(x) = f(x)$ whp.
2. *Strong model:* $f_{\mathrm{sab,str}} : (x, y, z_{x,y,*/\dagger}) \mapsto */\dagger$.
3. *Algorithm for $f_{\mathrm{sab,str}}$:* $\mathcal{B}$: Replace oracles in $\mathcal{A}$ with:
   1. Query z_j.
   2. If $z_j \in \{0,1\}$, return z_j $(= x_j = y_j)$.
   3. Else if $z_j = *$, return x_j.
   4. Else if $z_j = \dagger$, return y_j.



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   2. for any $x \in \{0,1\}^n$, $\mathcal{A}(x) = f(x)$ whp.
2. *Strong model:* $f_{\mathrm{sab,str}} : (x, y, z_{x,y,*/\dagger}) \mapsto */\dagger$.
3. *Algorithm for $f_{\mathrm{sab,str}}$:* $\mathcal{B}$: Replace oracles in $\mathcal{A}$ with:
   1. Query z_j.
   2. If $z_j \in \{0,1\}$, return z_j $(= x_j = y_j)$.
   3. Else if $z_j = *$, return x_j.
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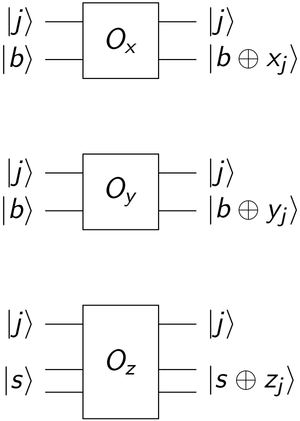
4. *Observation:*
   1. If z is a $*$-input, $\mathcal{B}$ feeds x into $\mathcal{A}$ $\Rightarrow$ $\mathcal{B}(\cdot) = \mathcal{A}(x) = 0$ whp.
   2. If z is a $\dagger$-input, $\mathcal{B}$ feeds y into $\mathcal{A}$ $\Rightarrow$ $\mathcal{B}(\cdot) = \mathcal{A}(y) = 1$ whp.



Algorithmic relation I: $\mathsf{QS}_{\mathsf{str}} = O(\mathsf{Q})$

Boolean function: $f : \{0,1\}^n \to \{0,1\}$.

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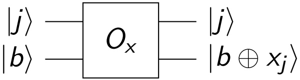

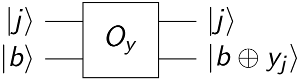

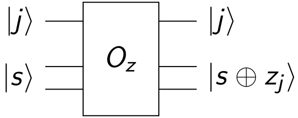

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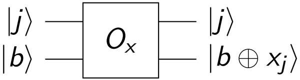

2. *Strong model:* $f_{\mathsf{sab,str}} : (x, y, z_{x,y,*/\dagger}) \mapsto */\dagger$.
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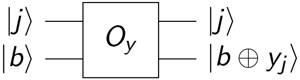

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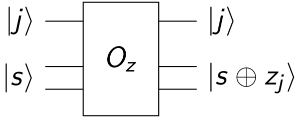

5. $\Rightarrow$ New algorithm computes $f_{\mathsf{sab,str}}$.

Conclusion: $\mathsf{QS}_{\mathsf{str}} = O(\mathsf{Q})$.

Algorithmic relation II: $\mathsf{QS}_{\mathsf{str}}^{\mathsf{ind}} = O(\mathsf{QD}^{3/2})$ (1/2)

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1. makes T queries.
2. on input x outputs $\left|\psi_x^{T+1}\right\rangle$ s.t.
   $f(x) \neq f(y) \Rightarrow |\left\langle \psi_x^{T+1} \middle| \psi_y^{T+1} \right\rangle| \leq 1/6$.

Quantum distinguishing complexity: min. T.

$$|j\rangle \; — \boxed{O_x} — \; (-1)^{x_j} |j\rangle$$

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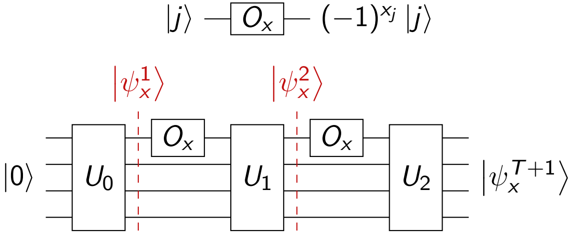

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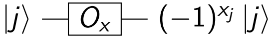

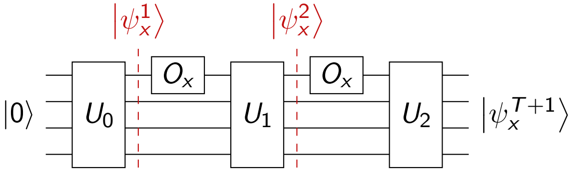

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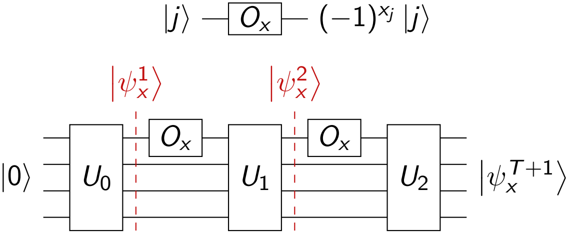

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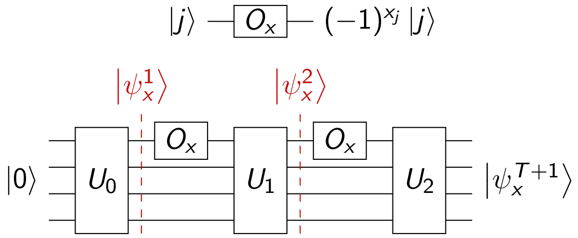
Intuition: if we run $\mathcal{A}$ on x and stop just before the t^{th} query, measuring gives $j \in J_z$ w.p. $p_{x,t}$.



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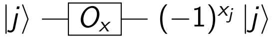

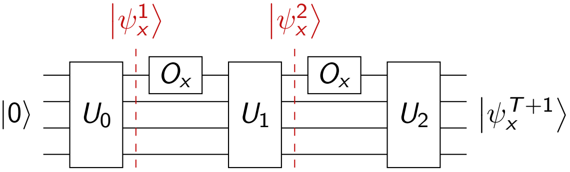

Lemma: $\sum_{t=1}^{T} p_{x,t} + p_{y,t} = \Omega(1)$.

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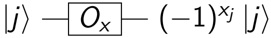

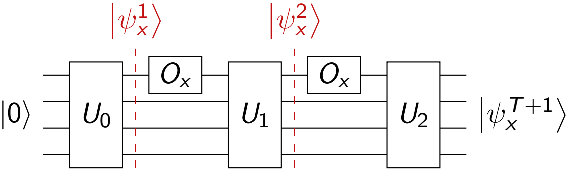

Lemma: $\sum_{t=1}^T p_{x,t} + p_{y,t} = \Omega(1)$.
Proof:
$$\left|\left\langle \psi_x^t \middle| \psi_y^t \right\rangle\right| - \left|\left\langle \psi_x^{t+1} \middle| \psi_y^{t+1} \right\rangle\right|$$
$$\leq \left|\langle \psi_x^t | (I - O_x^\dagger O_y) |\psi_y^t\rangle\right| \quad \text{(triangle ineq.)}$$
$$= 2\left|\langle \psi_x^t | \Pi_{x\neq y} |\psi_y^t\rangle\right|$$
$$\leq 2 \left\|\Pi_{x\neq y}|\psi_x^t\rangle\right\| \cdot \left\|\Pi_{x\neq y}|\psi_y^t\rangle\right\| \quad \text{(CS)}$$
$$\leq p_{x,t} + p_{y,t}. \quad \text{(AM-GM)}$$

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Algorithm: $\mathcal{B}$:

1. Select $t \in \{1, \dots, T\}$ u.a.r.
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Cost: $O(T)$ queries.
Success prob.: $\frac{1}{2T} \sum_{t=1}^{T} p_{x,t} + p_{y,t} = \Omega(\frac{1}{T})$.

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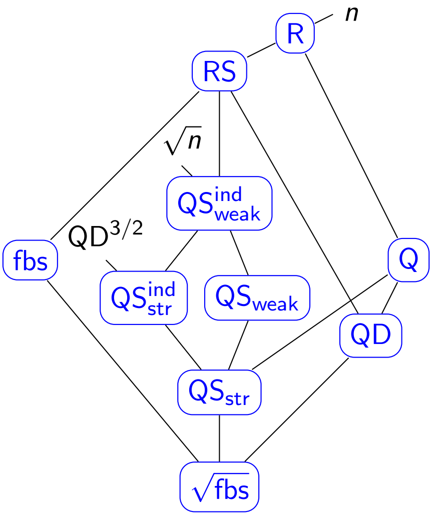

Separation: $\exists f : \mathrm{QS}_{\mathrm{str}}(f) = \Omega(\mathrm{fbs}(f))$

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1. *Indexing function:*
   1. $f : \{0,1\}^n \times \{0,1\}^{2^n} \to \{0,1\}$.
   2. $f(\ell, d) = d_\ell$.

$$x = \overbrace{\underbrace{001101}_{=\ell}}^{n \text{ bits}} \overbrace{01101 \cdots 0\,\underset{\displaystyle\downarrow\atop d_\ell}{1}\,0 \cdots 011}^{2^n \text{ bits}}$$

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   2. *Lemma:* For $R \subseteq X \times Y$, with
      1. $X \subseteq f_{\mathrm{sab,str}}^{-1}(*)$, $Y \subseteq f_{\mathrm{sab,str}}^{-1}(\dagger)$,
      2. $m_X = \max_{x \in X} |\{y : (x,y) \in R\}|$
      3. $m_Y = \max_{y \in Y} |\{x : (x,y) \in R\}|$
      4. $\ell_{\max} = \max_{x \in X, y \in Y, j \in [n+2^n]}$
         $|\{y' : (x,y') \in R, x_j \neq y_j\}|$
         $\cdot |\{x' : (x',y) \in R, x_j \neq y_j\}|$

      $\Rightarrow \mathrm{QS}_{\mathrm{str}} = \Omega\left(\sqrt{\frac{m_X m_Y}{\ell_{\max}}}\right)$. [Amb02]

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      1. $X \subseteq f_{\mathrm{sab,str}}^{-1}(*)$, $Y \subseteq f_{\mathrm{sab,str}}^{-1}(\dagger)$,
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         $|\{y' : (x,y') \in R, x_j \neq y_j\}|$
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   $\Rightarrow \mathrm{QS}_{\mathrm{str}} = \Omega\left(\sqrt{\frac{m_X m_Y}{\ell_{\max}}}\right)$. [Amb02]

$$x = \overbrace{\underbrace{001101}_{=\ell}}^{n \text{ bits}}\ \overbrace{01101\cdots 0\,\underset{\substack{\downarrow \\ d_\ell}}{1}\,0\cdots 011}^{2^n \text{ bits}}$$

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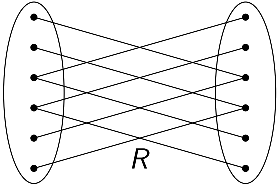

Separation: $\exists f : \mathrm{QS}_{\mathrm{str}}(f) = \Omega(\mathrm{fbs}(f))$

1. *Indexing function:*
   1. $f : \{0,1\}^n \times \{0,1\}^{2^n} \to \{0,1\}$.
   2. $f(\ell, d) = d_\ell$.
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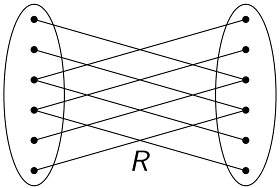

ℓ	ℓ'
$\vdots$	$\vdots$
$100\ 000{*}0000$	$100\ 000\dagger0000$
$101\ 0000{*}000$	$101\ 0000\dagger000$
$\vdots$	$\vdots$

$|\ell \oplus \ell'| = 2$

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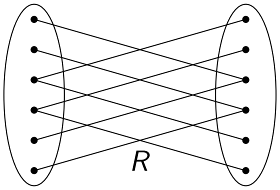

ℓ	$\vdots$	ℓ'	$\vdots$
100 000$*$0000		100 000$\dagger$0000	
101 0000$*$000		101 0000$\dagger$000	
$\vdots$	$|\ell \oplus \ell'| = 2$	$\vdots$	

Summary & open questions

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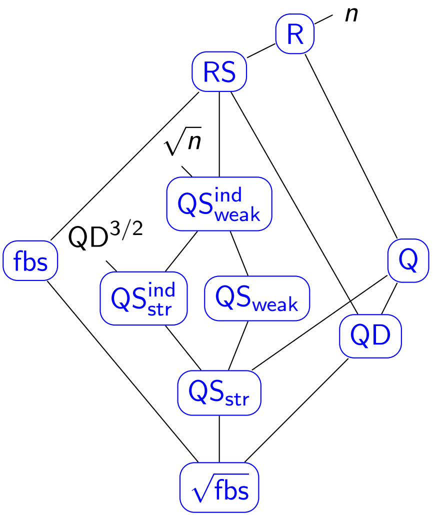

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Open questions:

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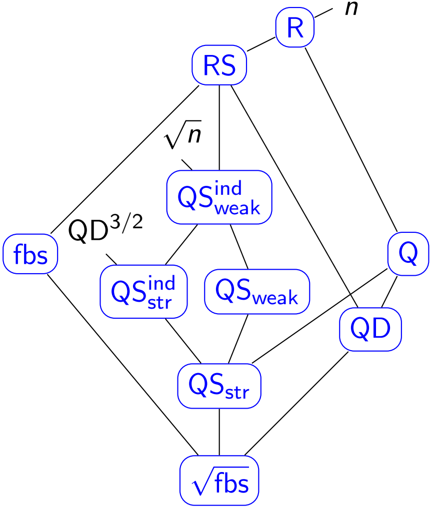

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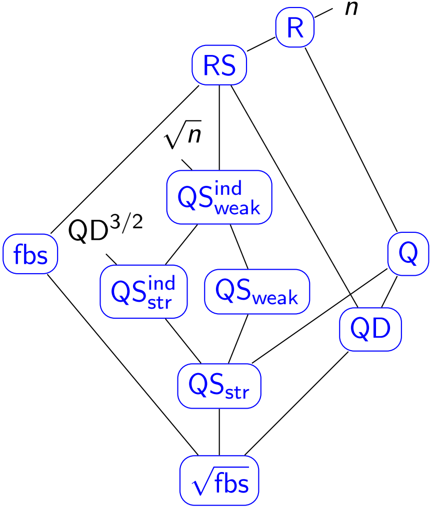

Thanks for your attention!

ajcornelissen@outlook.com