Quantum algorithms through graph composition

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SIMONS INSTITUTE for the Theory of Computing

Quantum algorithmic frameworks

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Setting:

1. $f : \mathcal{D} \to \{0,1\}$.
2. $\mathcal{D} \subseteq \{0,1\}^n$.
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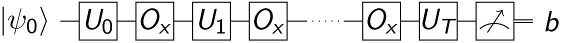

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3. With minimal no. queries T.

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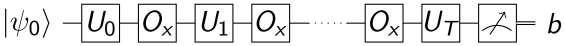

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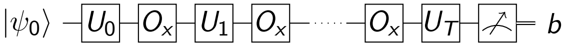

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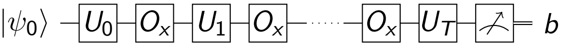

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[Rei11]

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$$|\psi_0\rangle \; \underbrace{U_0 \; O_x \; U_1 \; O_x \cdots O_x \; U_T \; \text{measure}}_{T = O(\sqrt{\ell})} = b$$

Quantum algorithmic framework landscape

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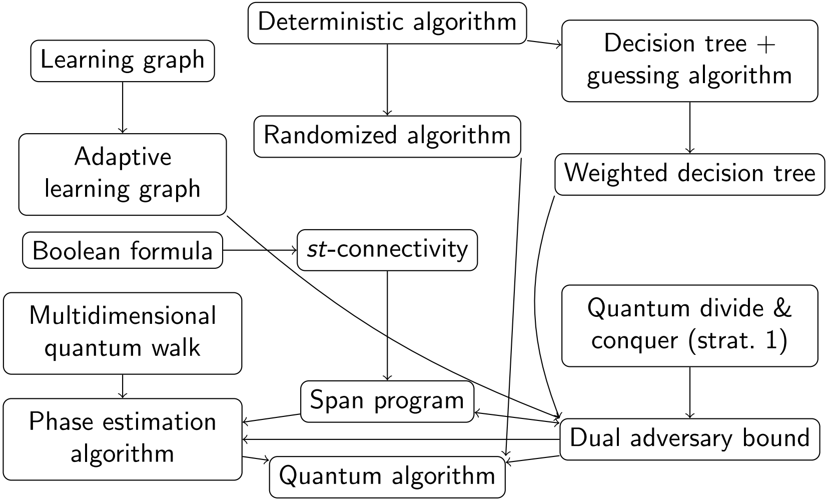
Overview:

→ Conversion between framework objects



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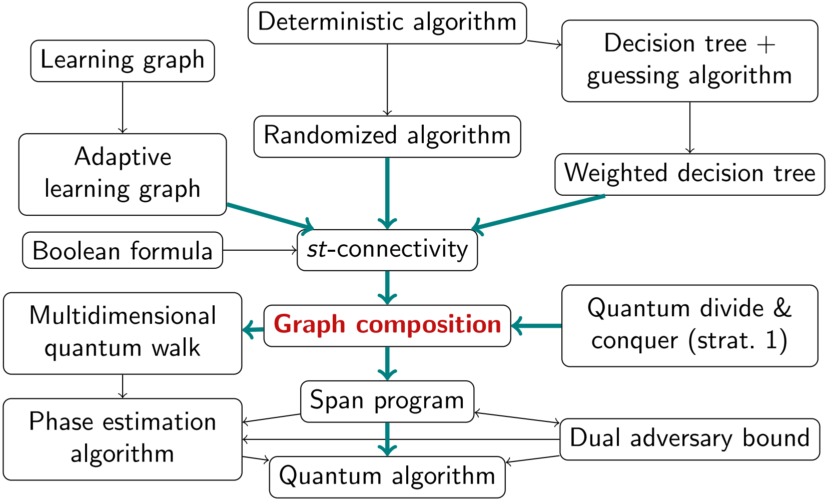

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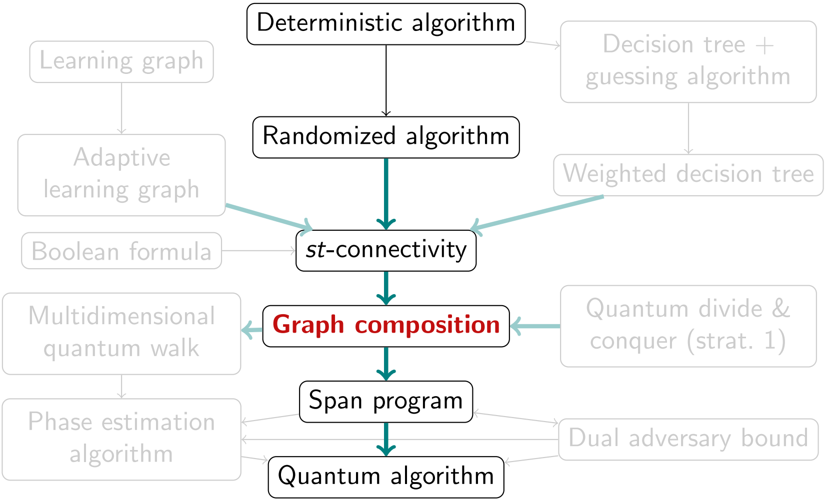
This talk:

1. Span programs
2. Graph composition
3. *st*-connectivity examples
4. Randomized → *st*-connectivity



Span programs [RŠ12,Rei09,Rei11]

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Four ingredients:

1. *Hilbert space:* $\mathcal{H}$.
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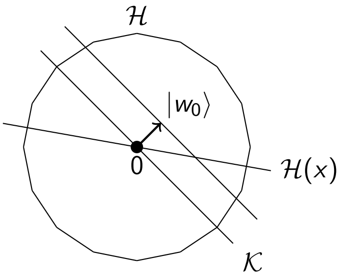

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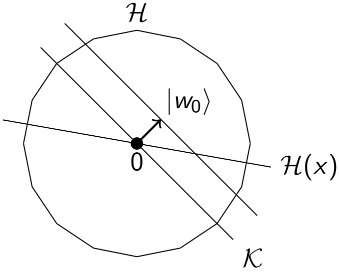

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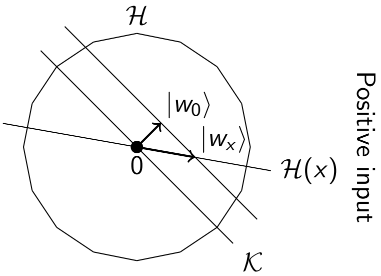

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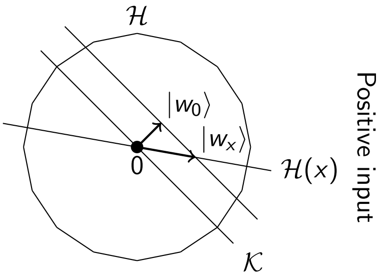

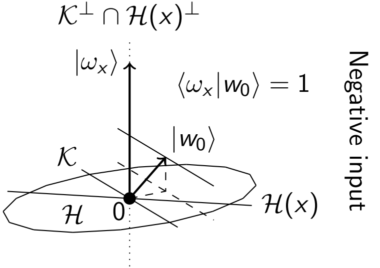

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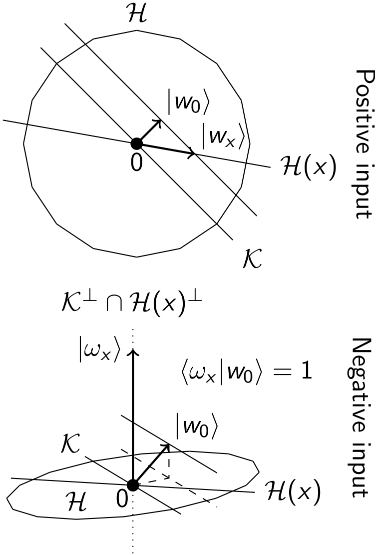

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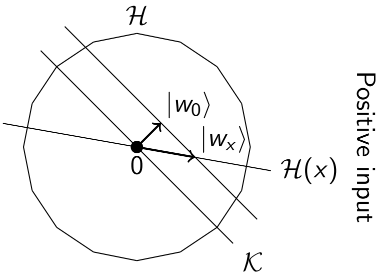
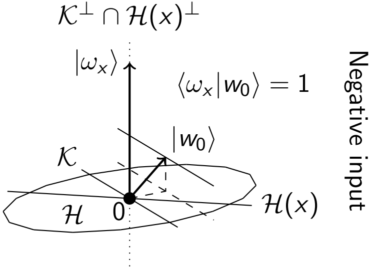
Thm: "Complete" for computing boolean functions.





Graph composition [This work]

Ingredients:

1. Undirected graph $G = (V, E)$.
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3. Edge span programs $(\mathcal{P}_e)_{e \in E}$ on $\mathcal{D}$.

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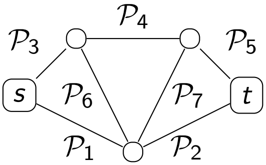

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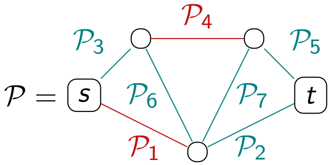

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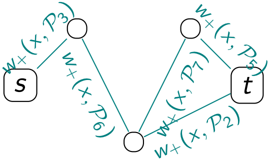
$\mathcal{P} =$ 

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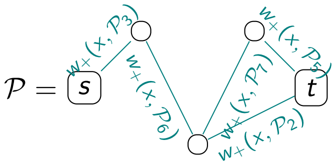

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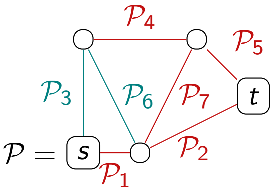

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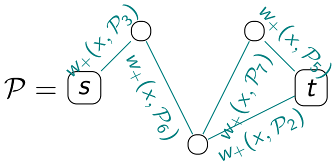

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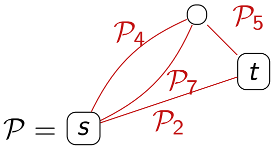

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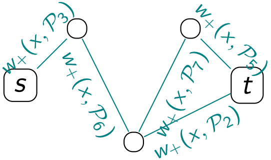
$\mathcal{P} =$ 

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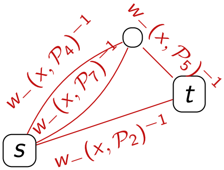
$\mathcal{P} =$ 

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Path-cut theorem:

1. For P an st-path: $w_+(x, \mathcal{P}) \leq \sum_{e \in P} w_+(x, \mathcal{P}_e)$.
2. For C an st-cut: $w_-(x, \mathcal{P}) \leq \sum_{e \in C} w_-(x, \mathcal{P}_e)$.

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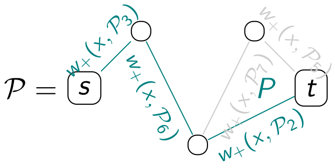

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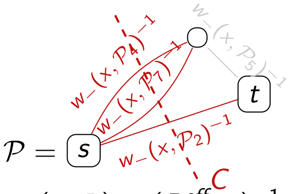

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For $\alpha > 0$, $j \in [n]$:

αx_j

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Example: The OR-function

1. $\mathrm{OR}_n : \{0,1\}^n \to \{0,1\}$

$$\mathrm{OR}_n(x) = \begin{cases} 1, & \text{if } |x| \geq 1, \\ 0, & \text{if } |x| = 0. \end{cases}$$

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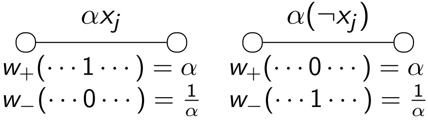
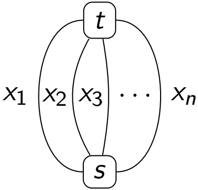


st-graph for OR_n*:*



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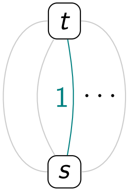

$x = 0010\cdots 0 \Rightarrow w_+(x) \leq 1$

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1. *Trivial span programs:* single bit queries.

Example: The OR-function

1. $\mathrm{OR}_n : \{0,1\}^n \to \{0,1\}$
$$\mathrm{OR}_n(x) = \begin{cases} 1, & \text{if } |x| \geq 1, \\ 0, & \text{if } |x| = 0. \end{cases}$$
2. $w_+(x) \leq 1$.
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For $\alpha > 0$, $j \in [n]$:

αx_j

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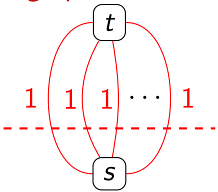
st-graph for OR_n*:*



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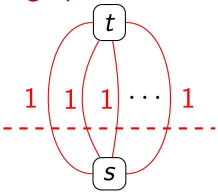

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Quadratic speed-up for search.

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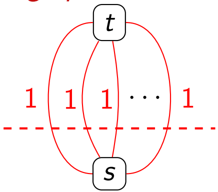

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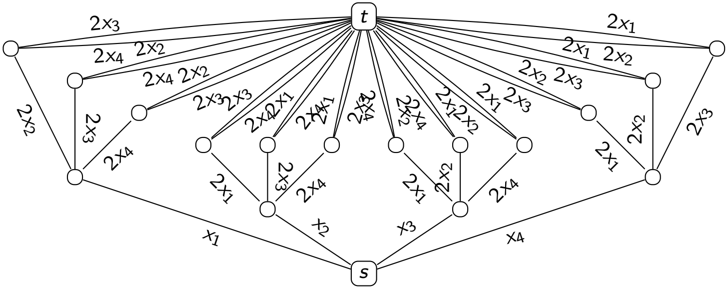

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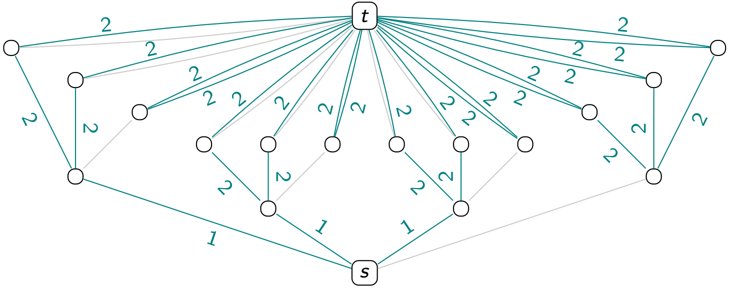

$x = 1110 \Rightarrow w_+(x) = 1$

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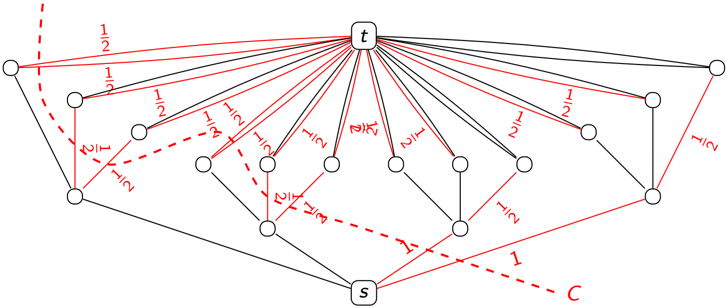

$x = 1100 \Rightarrow w_-(x) = 6$

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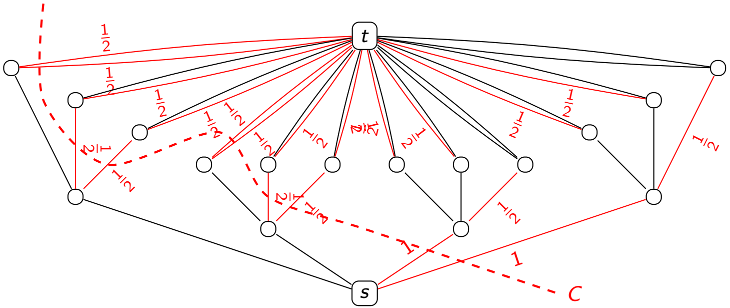

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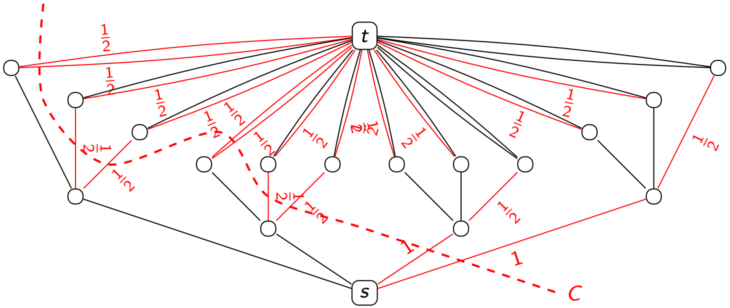

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Remark: With $k = n/2$, it also solves gapped majority in $O(1)$ queries.

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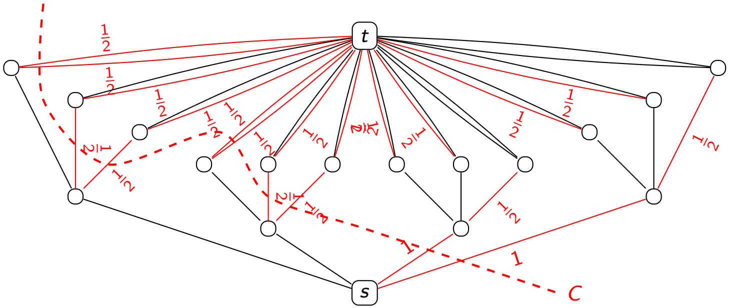

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Classical algorithms → st-connectivity

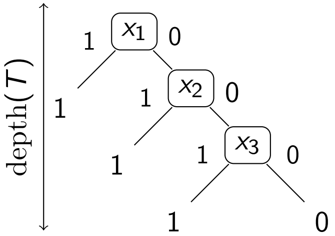
Classical algorithms → *st*-connectivity

Deterministic → st-connectivity:

Classical algorithms $\to$ *st*-connectivity

Deterministic $\to$ st-connectivity:

1. Decision tree T.



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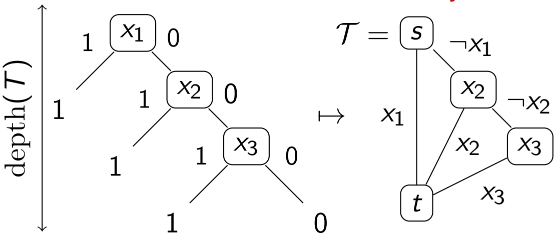

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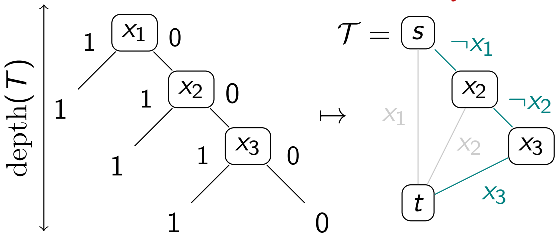

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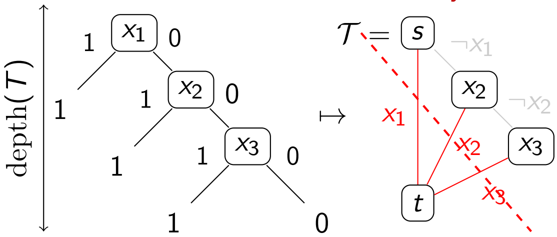

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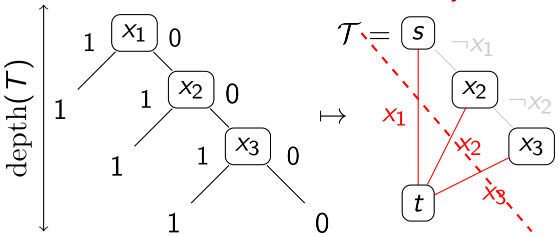

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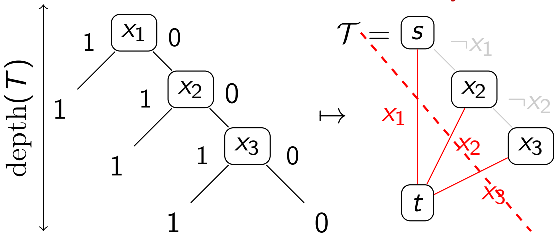

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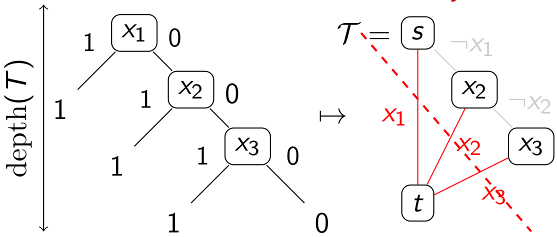

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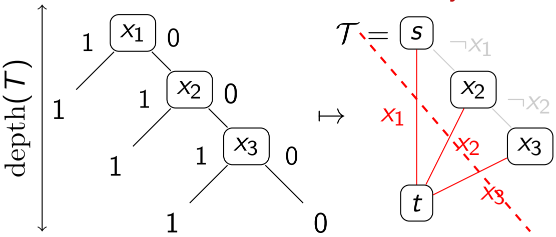

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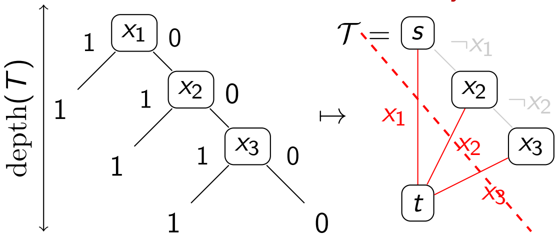

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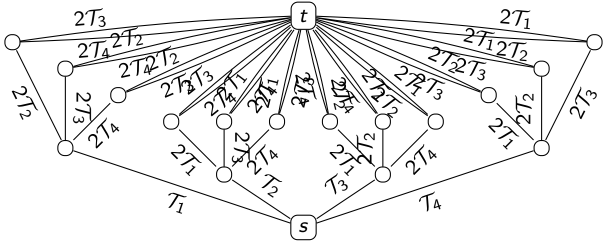

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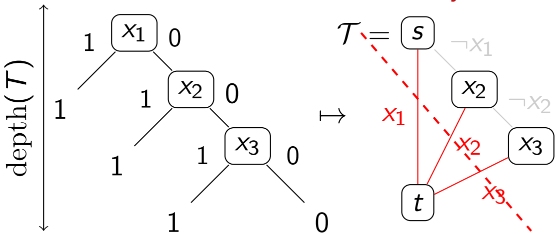

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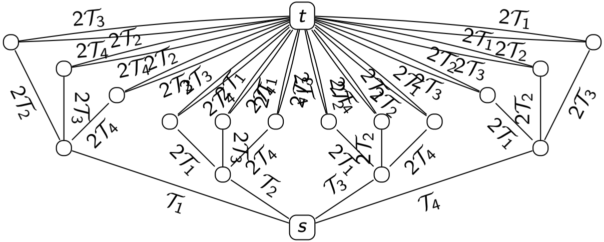

Summary

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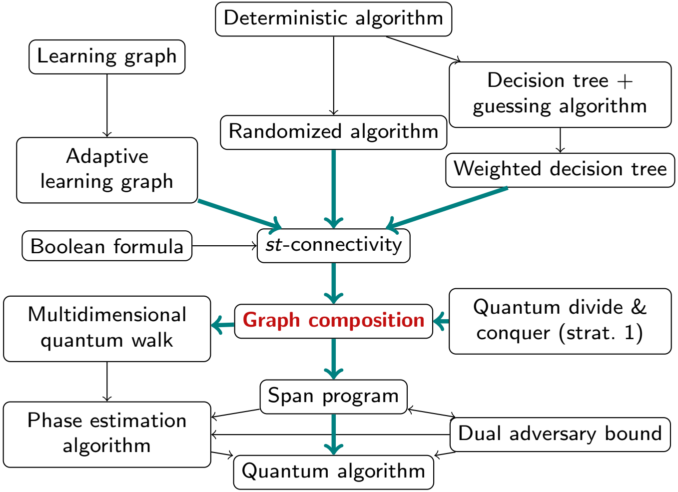
Relations between algorithmic frameworks



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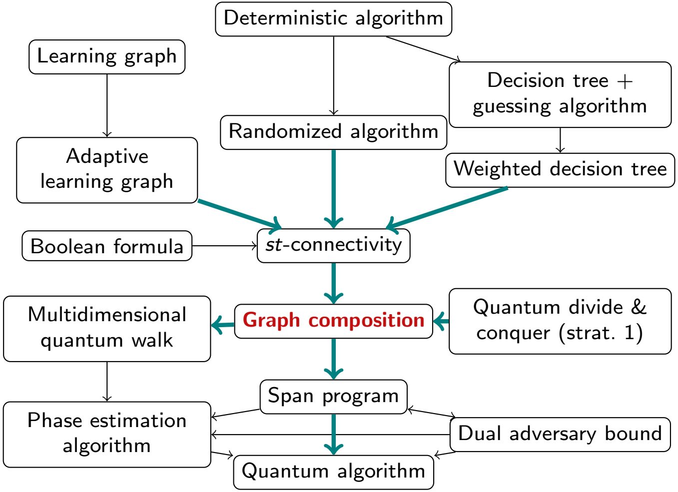

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2. Graph composition
3. Examples
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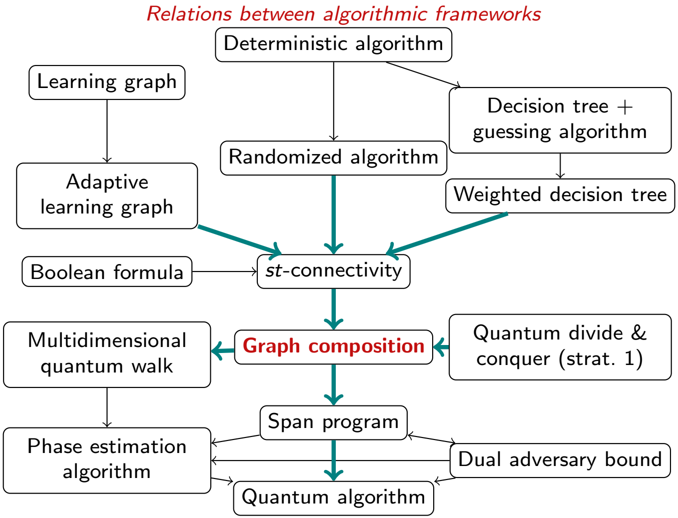

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4. More examples

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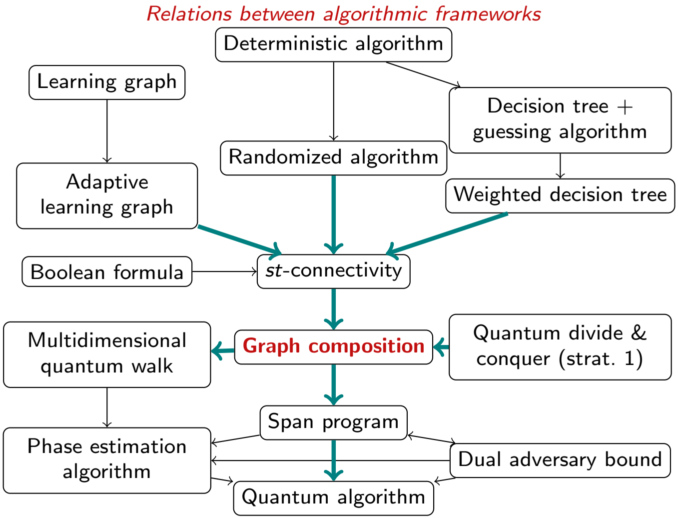

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Open question:

Limitations of *st*-connectivity?

Thanks!

Thanks for your attention!

ajcornelissen@outlook.com

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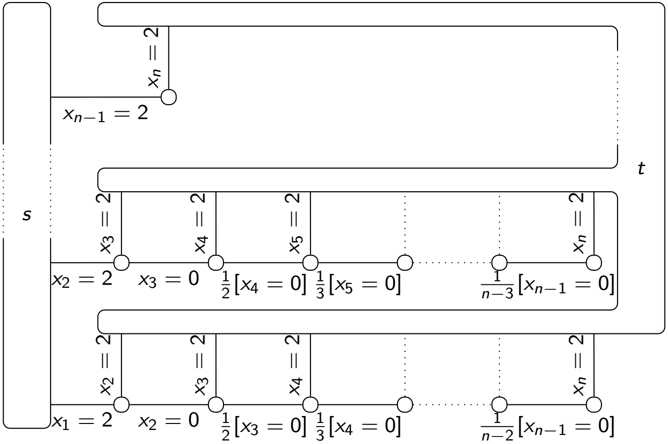

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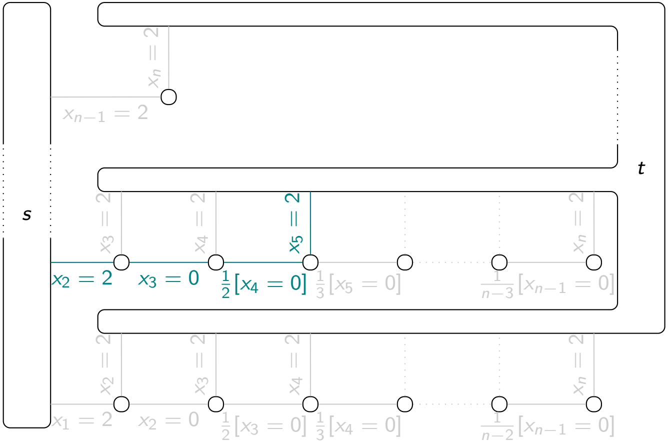

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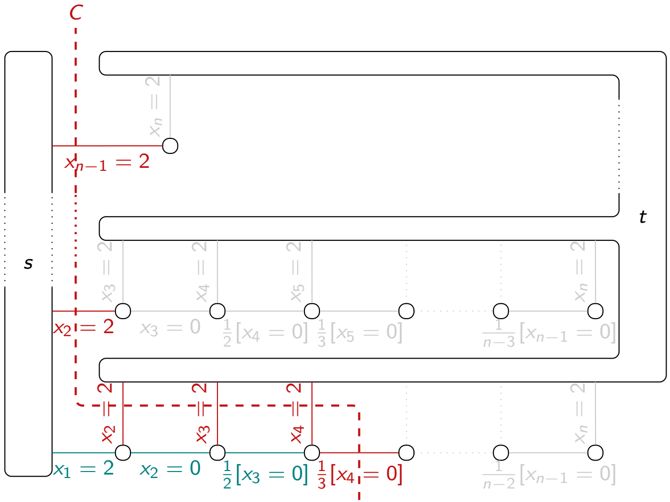

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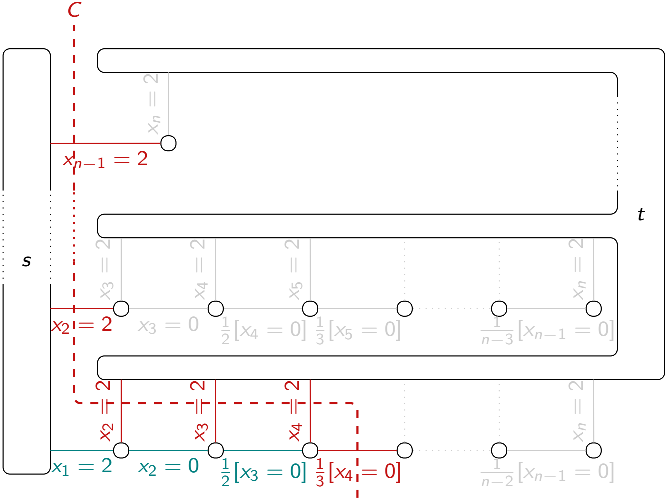
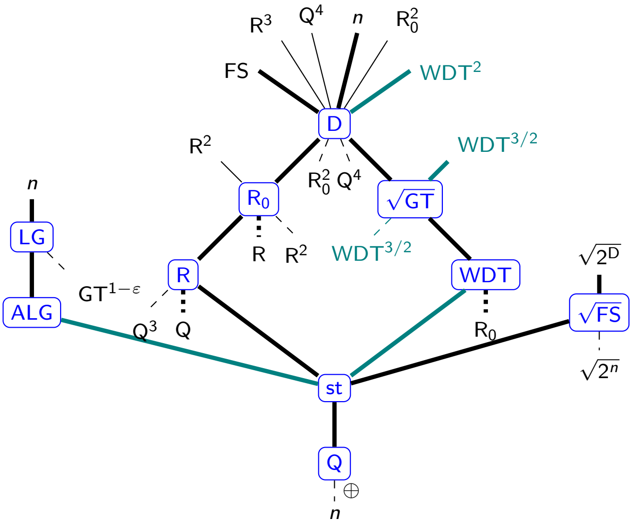


Complexity measure relations for total boolean functions



Legend:

1. Solid: relation.
2. Dashed: separation.
3. **Fat:** for partial functions.
4. Dotted: unbounded separation.
5. Green: new in this work.

Open questions:

1. Unbounded separation:
   1. Between Q and st?
   2. Between st and R?