Quantum algorithms through composition of graphs

Arjan Cornelissen[1]

[1]Simons Institute, University of California, Berkeley, California

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Quantum algorithms

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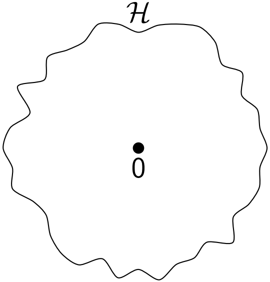

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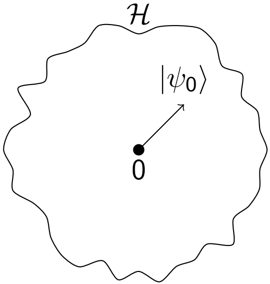

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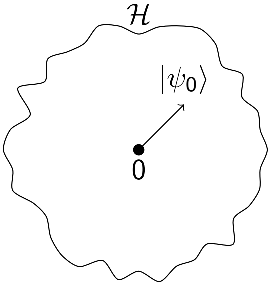

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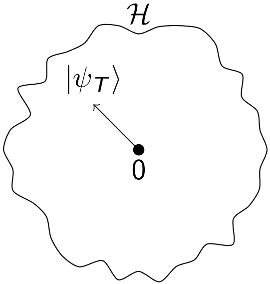

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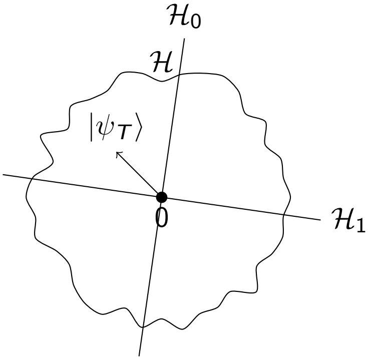

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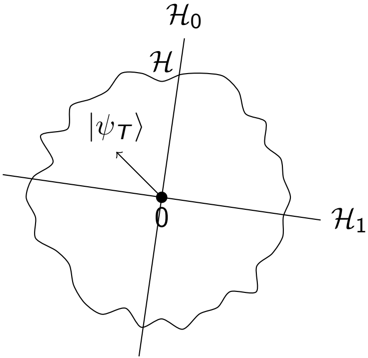

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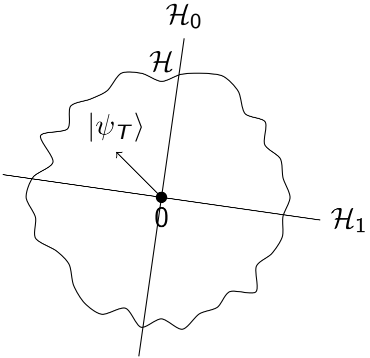

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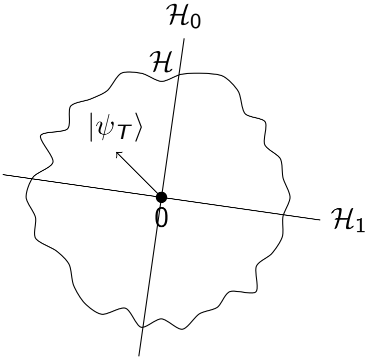

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Quantum query complexity: $\mathrm{Q}(f; O_x)$:

1. Minimum number of oracle calls.

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for boolean function f:

1. $f : \mathcal{D} \to \{0,1\}$.
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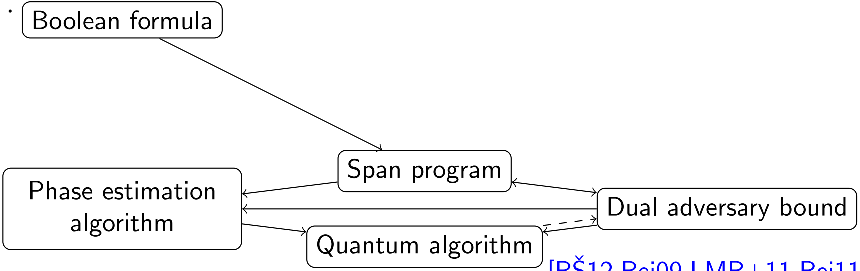

[RŠ12,Rei09,LMR+11,Rei11]

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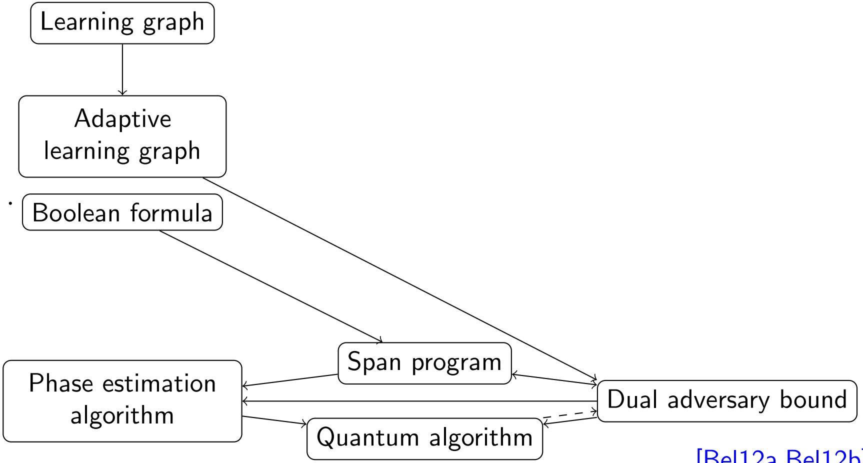

[Bel12a,Bel12b]

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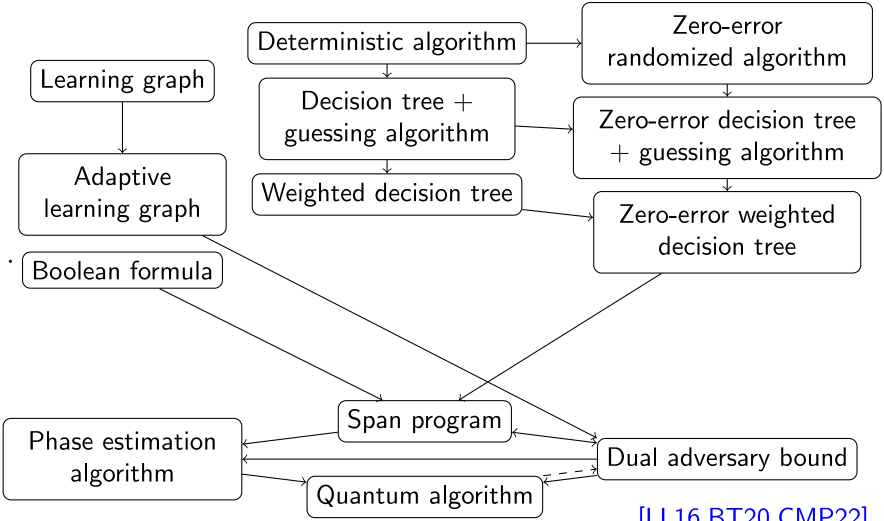

[LL16,BT20,CMP22]

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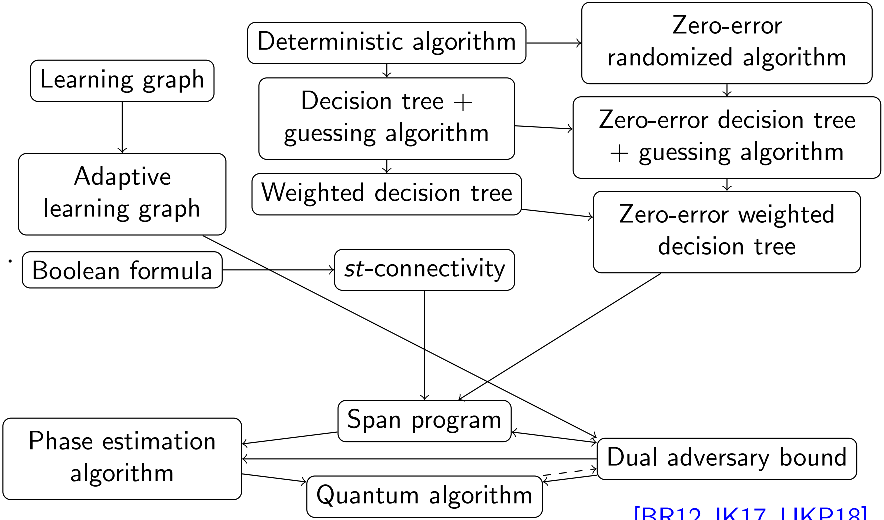

[BR12,JK17,JJKP18]

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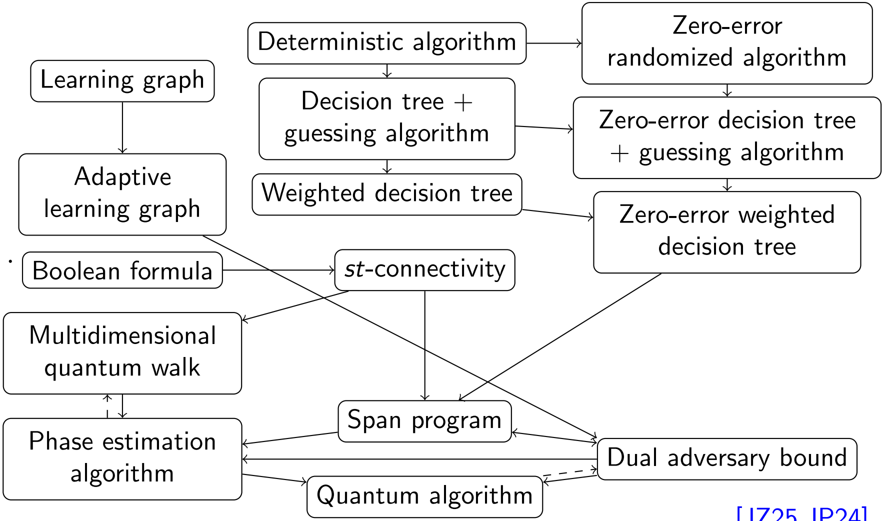

[JZ25,JP24]

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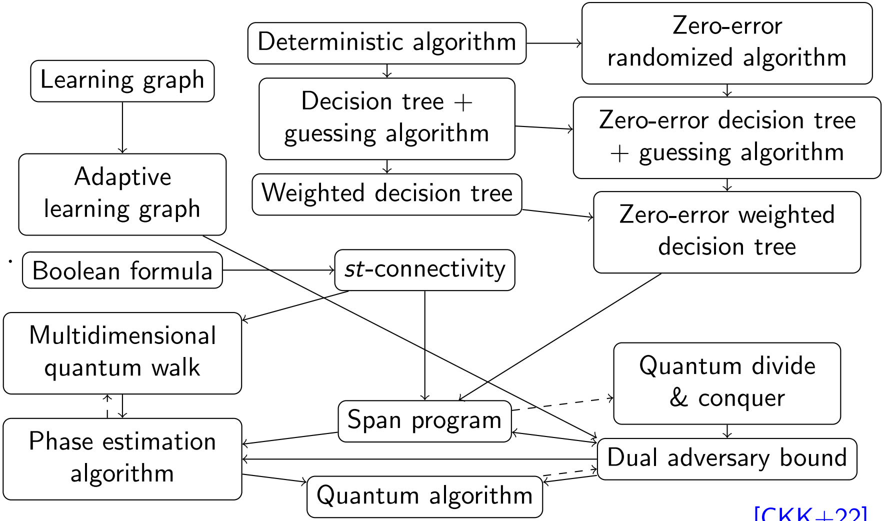

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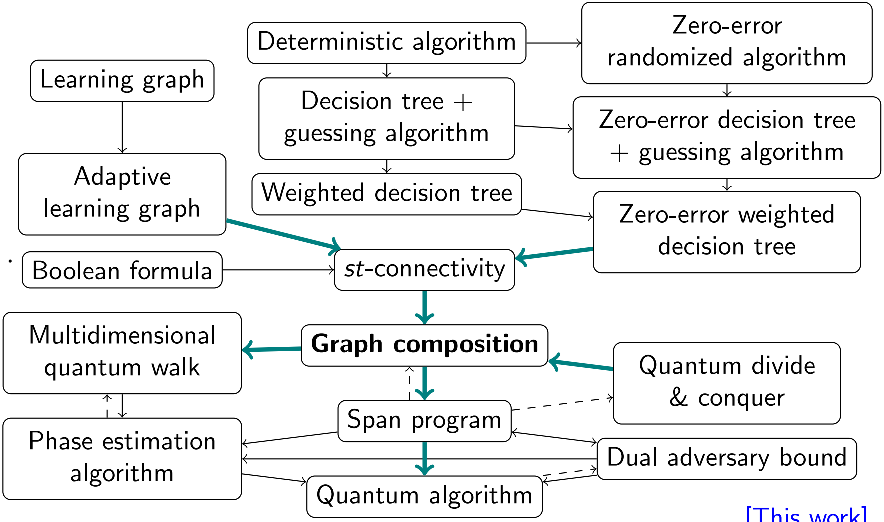

[This work]

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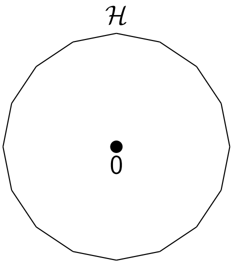

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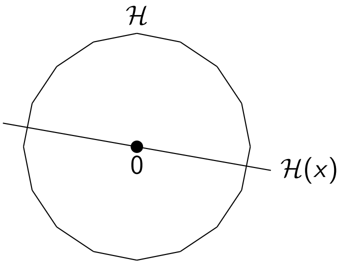

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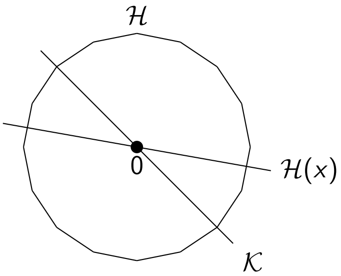

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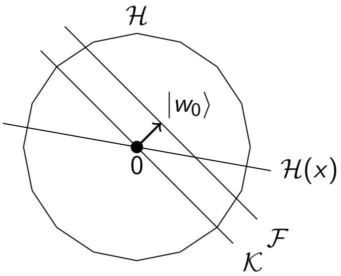

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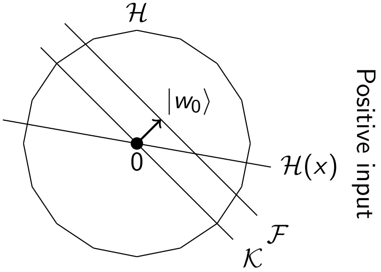

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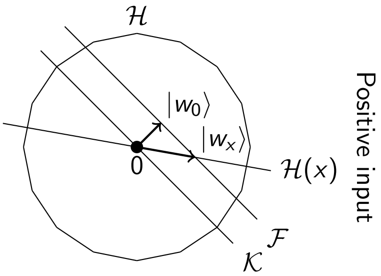

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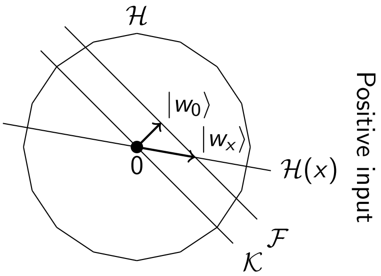

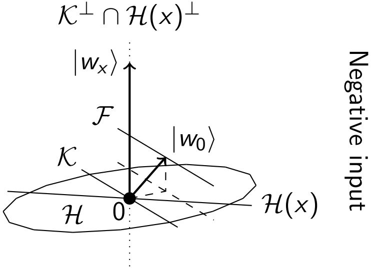

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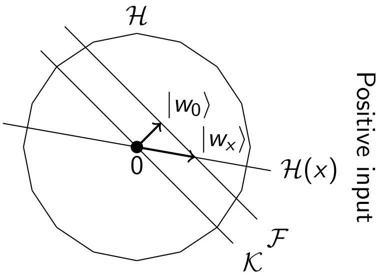
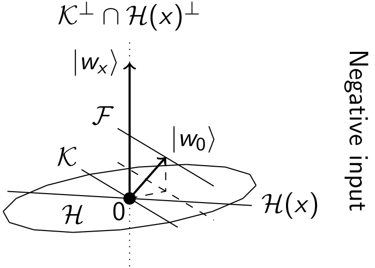
4. $C(\mathcal{P}) = \sqrt{\max_{x \in f^{-1}(0)} w_-(x,\mathcal{P}) \cdot \max_{x \in f^{-1}(1)} w_+(x,\mathcal{P})}$.





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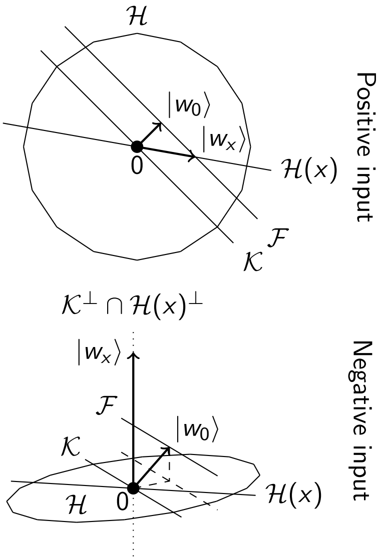
Thm: $Q(f; 2\Pi_{\mathcal{H}(x)} - I) = O(C(\mathcal{P}))$ [Rei11].



Electrical networks and span programs [BR12, JK17, JJKP18]

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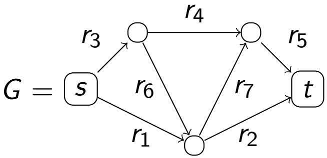
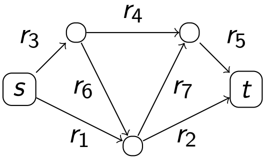
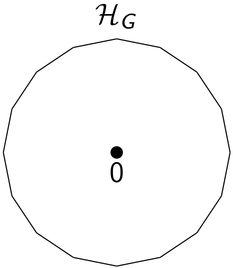


Graph $G = (V, E)$, resistances $r : E \to [0, \infty]$, $s, t \in V$.

1. *Flow:* $f : E \to \mathbb{C}$.
   Flow space: $\mathcal{H}_G = \mathrm{Span}\{|e\rangle : e \in E\}$,
   $f \mapsto |f_{G,r}\rangle = \sum_{e \in E} f_e \sqrt{r_e}\,|e\rangle$.

$G =$ 



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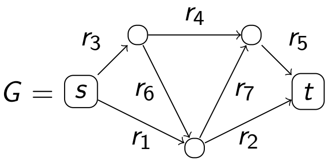

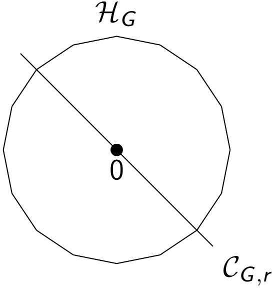

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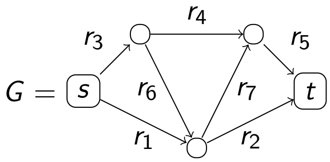

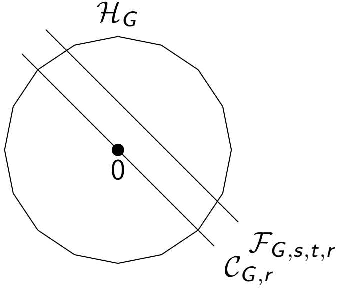

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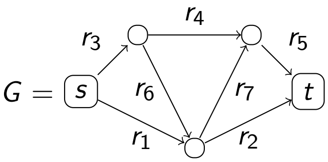

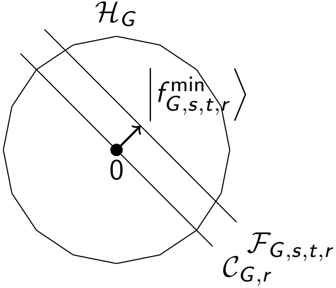

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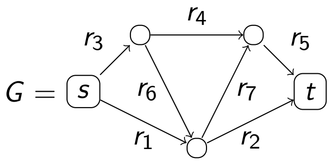

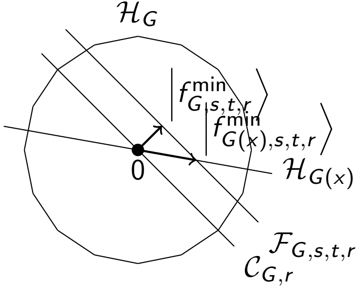

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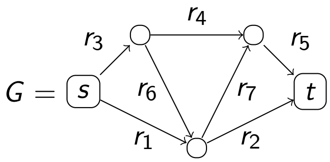

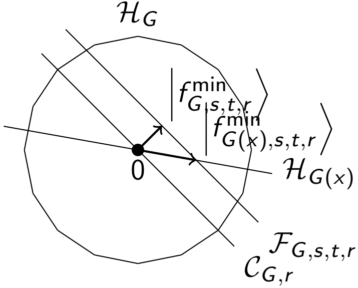

st-connectivity span program: $(\mathcal{H}_G, x \mapsto \mathcal{H}_{G(x)}, \mathcal{C}_{G,r}, |f^{\min}_{G,s,t,r}\rangle)$.

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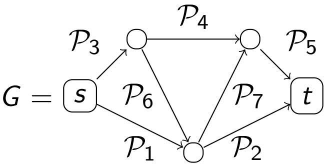

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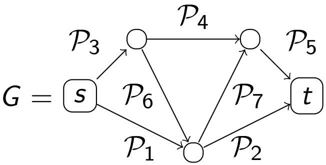

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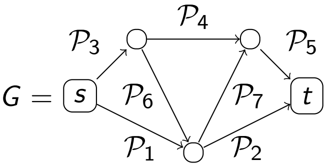

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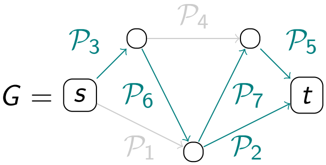

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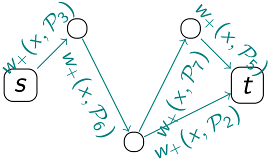

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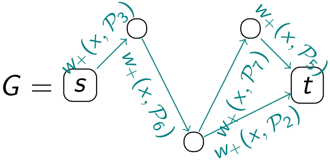

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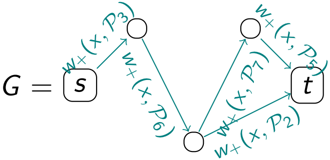

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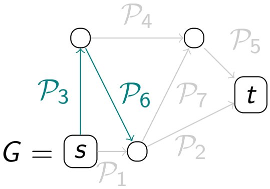

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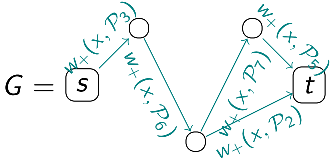

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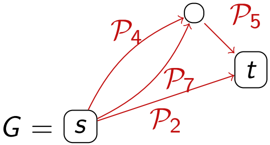

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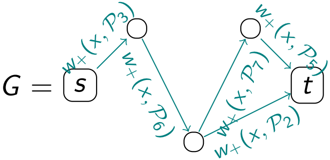

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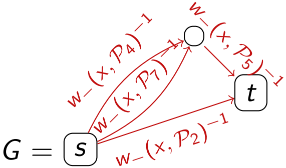

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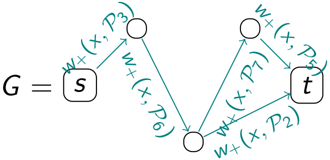

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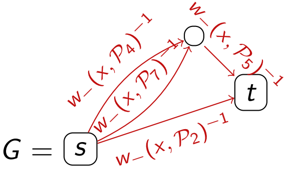

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Path-cut theorem

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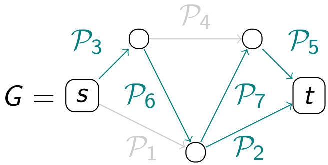
Positive input:



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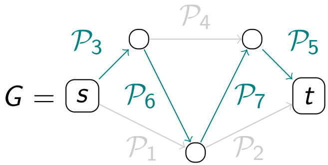

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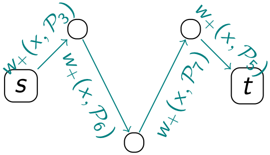
$G =$ 

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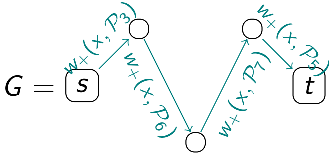

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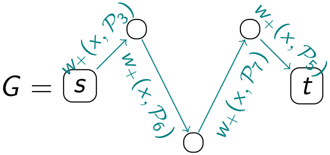

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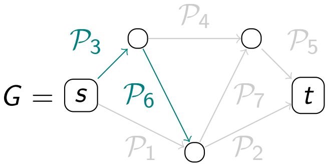
Negative input:



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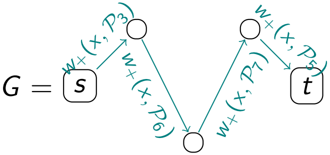

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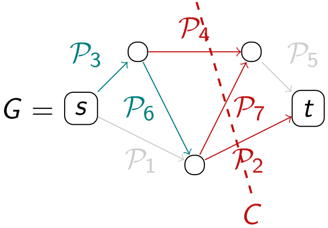

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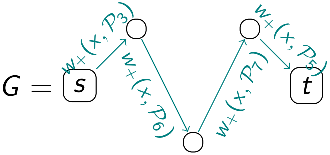

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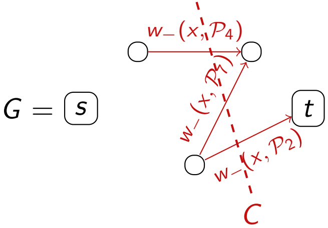

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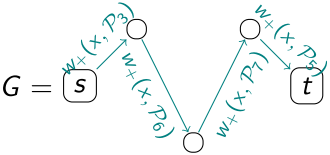

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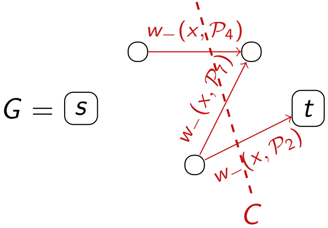

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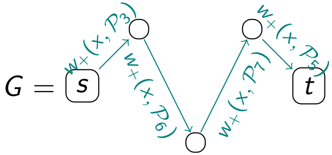
Properties:

1. Simpler (less-powerful) version.
2. Still powerful enough for many applications.

Positive input:



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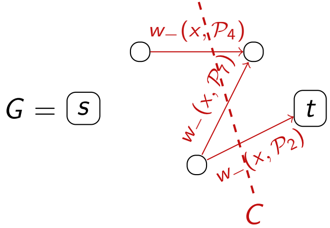

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Example: OR-function

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Trivial span program: $(\alpha > 0)$

1. $x_j : \{0,1\}^n \to \{0,1\}$, $x \mapsto x_j$.
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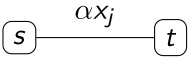

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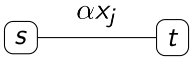

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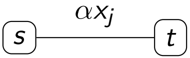

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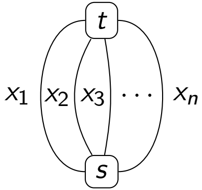
Graph composition for OR_n:



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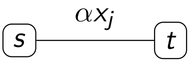

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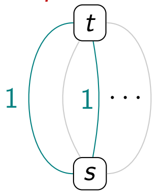

$x = 1010\cdots0 \Rightarrow w_+(x) = \frac{1}{2}$

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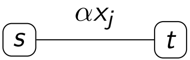

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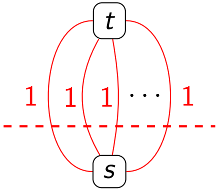

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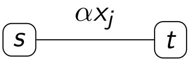

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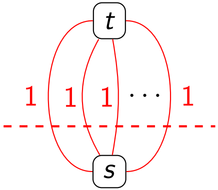

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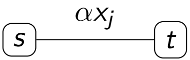
Trivial span program for x_j:



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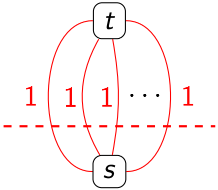

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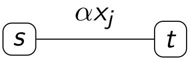
Quadratic speed-up for search.

Trivial span program for x_j:



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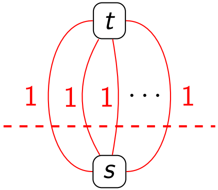

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Example: Threshold function

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The threshold function: $(k \in [n])$

1. $\mathrm{Th}_n^k : \{0,1\}^n \to \{0,1\}$
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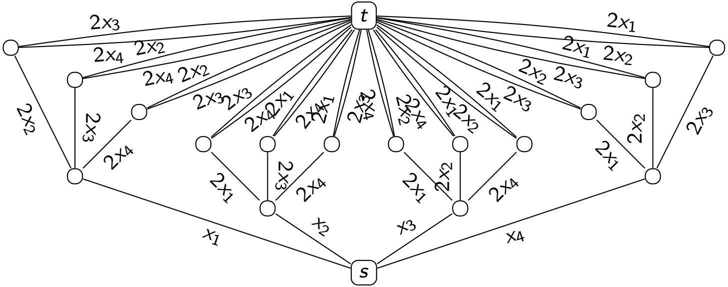
Graph composition for Th_4^3*:*



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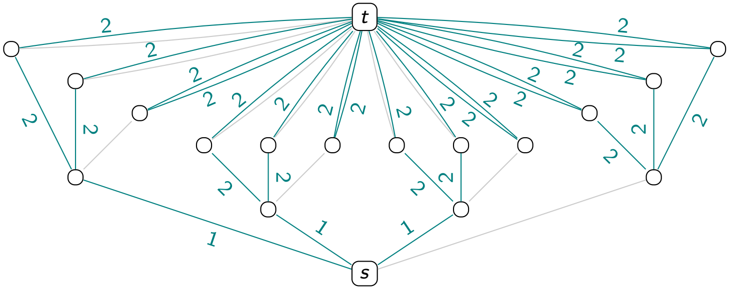

$x = 1110 \Rightarrow w_+(x) = 1$

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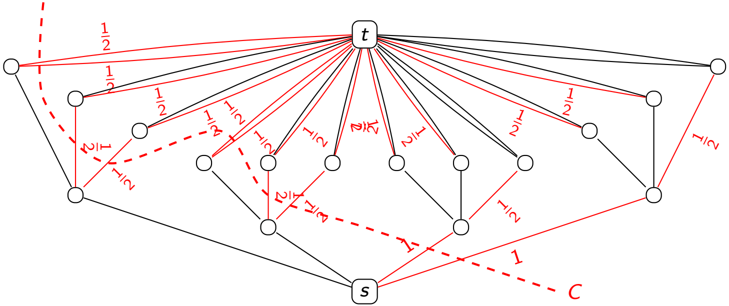

$x = 1100 \Rightarrow w_-(x) = 6$

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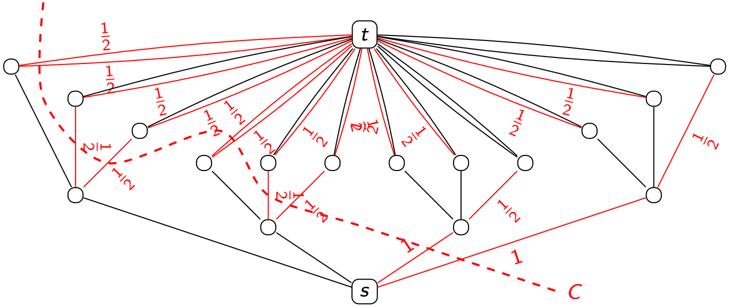

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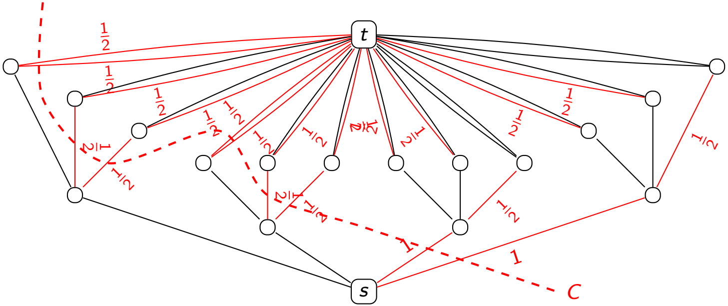

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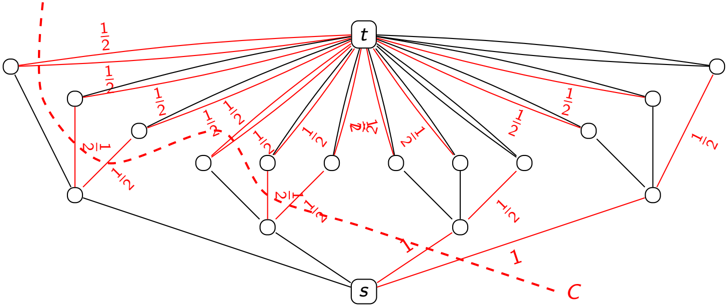
   $$\mathrm{Th}_n^k(x) = \begin{cases} 1, & \text{if } |x| \geq k, \\ 0, & \text{if } |x| < k. \end{cases}$$
2. $w_+(x) = \frac{1}{|x|-k+1}$
3. $w_-(x) = \frac{k(n-k+1)}{k-|x|}$
4. $C(\mathcal{P}) = \sqrt{k(n-k+1)}$.
5. $\Rightarrow Q(\mathrm{Th}_n^k) \in O(\sqrt{k(n-k+1)})$.

Known to be optimal!

Graph composition for Th_4^3*:*



$x = 1100 \Rightarrow w_-(x) = 6$

Example: the $\Sigma^*20^*2\Sigma^*$-problem

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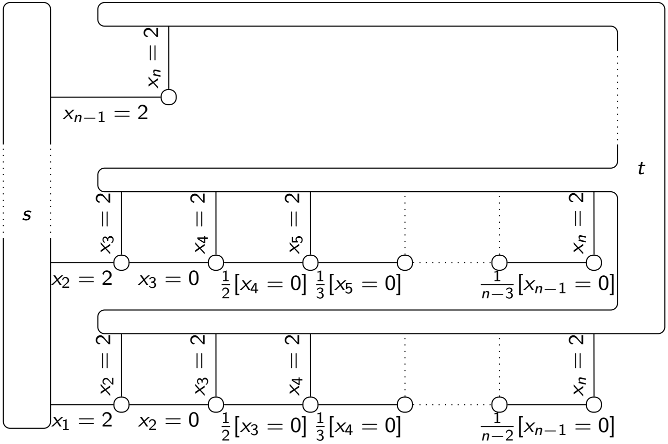

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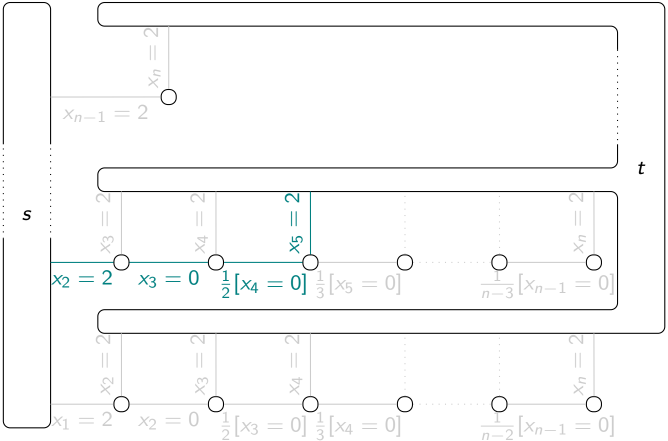

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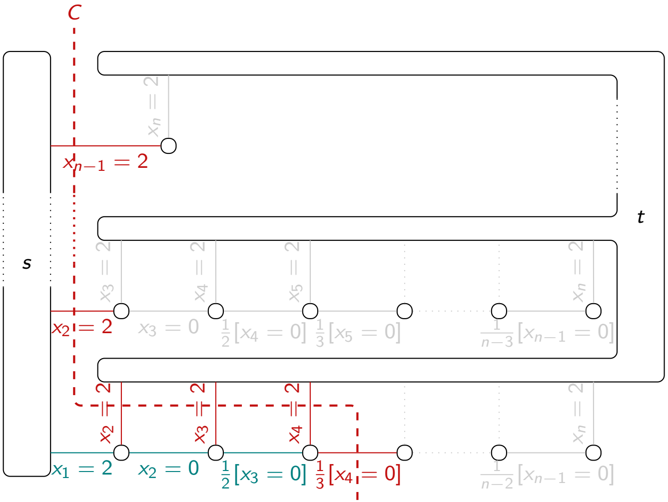

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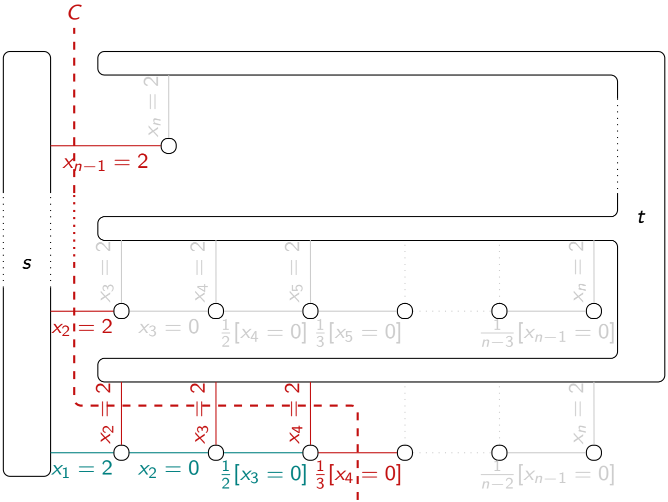
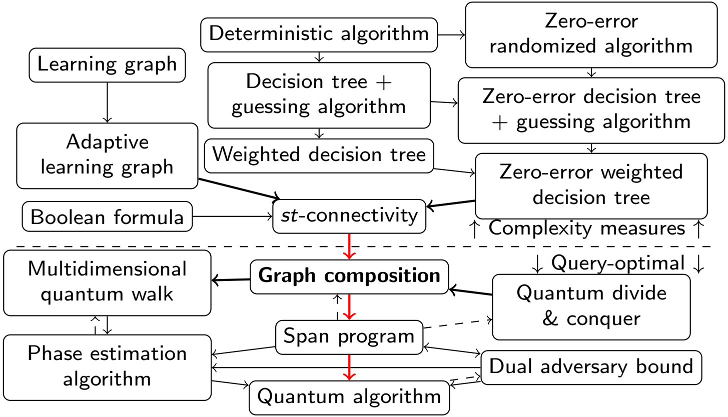


Summary (I/II)

Relations between algorithmic frameworks



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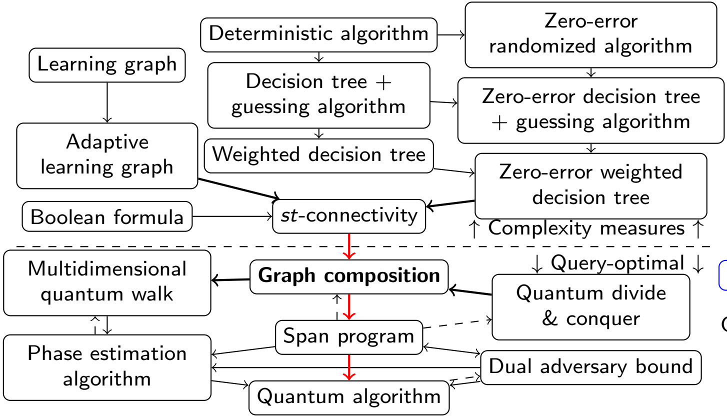
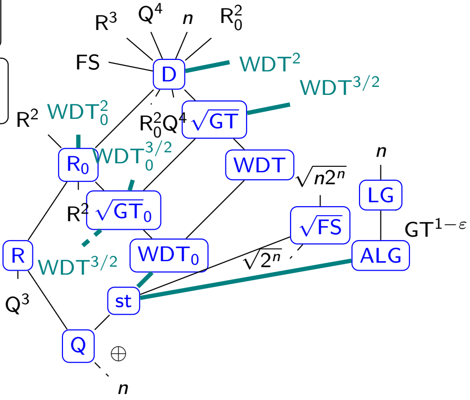


Relations between complexity measures



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Thanks for your attention!

`ajcornelissen@outlook.com`

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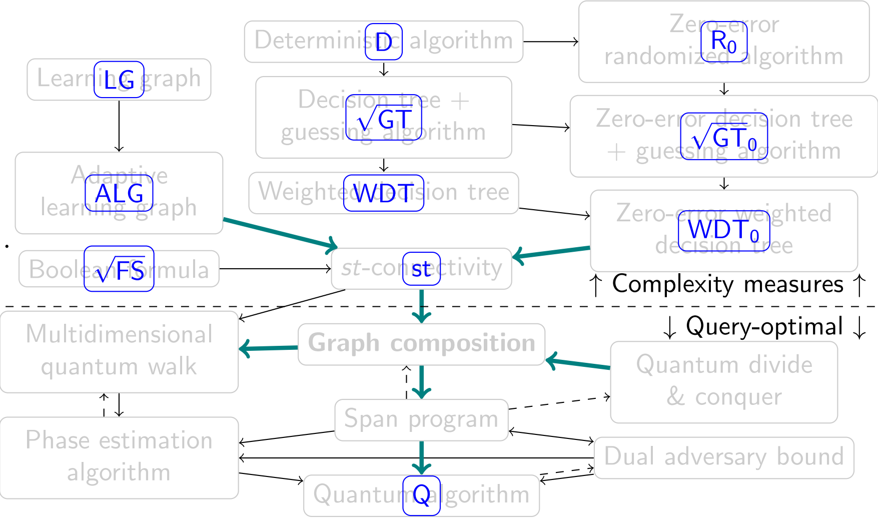
Quantum algorithmic frameworks (for boolean functions)

Goal: Design algorithm for boolean function f:

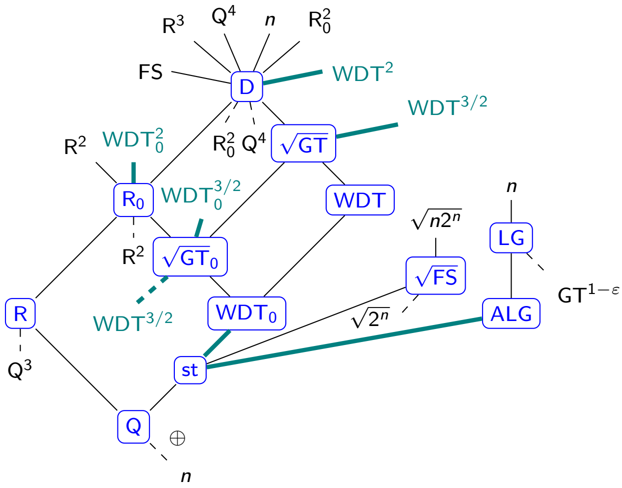
1. $f : \mathcal{D} \to \{0, 1\}$.
2. $\mathcal{D} \subseteq \{0, 1\}^n$.
3. $O_x : |j\rangle \mapsto (-1)^{x_j} |j\rangle$.

Framework:

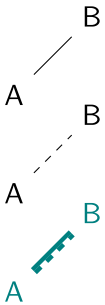
1. Define object L.
2. Convert object into quantum algorithm.



Complexity measure relations for total boolean functions



Legend:



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$A(f) \in \widetilde{O}(B(f))$

$\exists f : \{0,1\}^n \to \{0,1\}$
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New in this work

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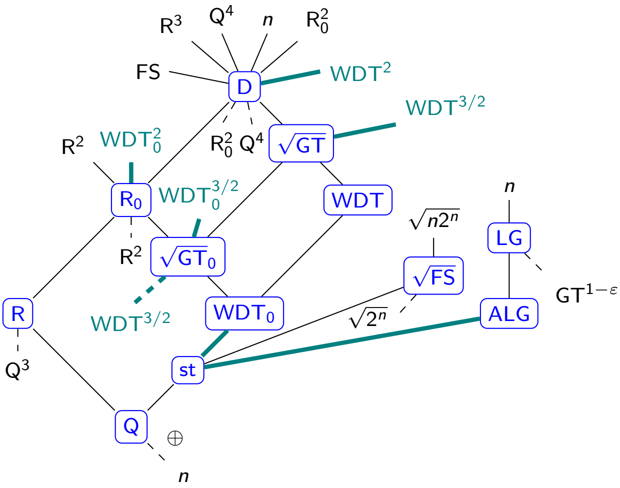

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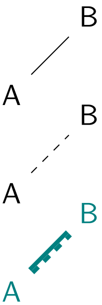

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Open questions:

1. Separation between Q and st?
2. Can we prove $R \in \widetilde{O}(st^2)$?