A sublinear time quantum algorithm for approximating partition functions

arXiv:2207.08643

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January 22nd, 2023

Research Center for Quantum Software

Counting independent sets

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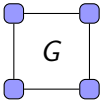

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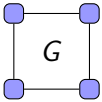

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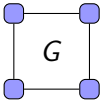

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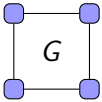

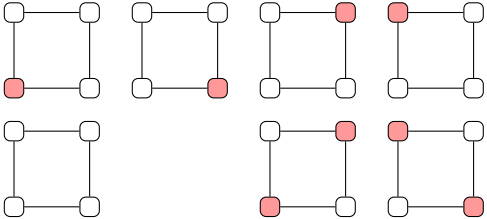

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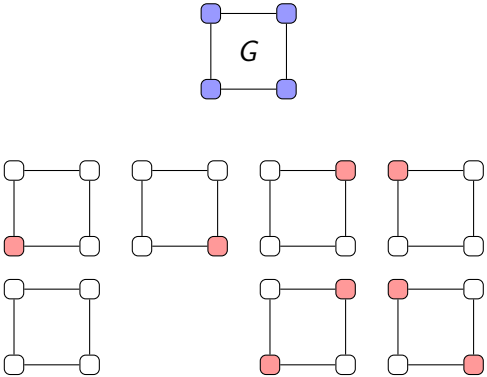

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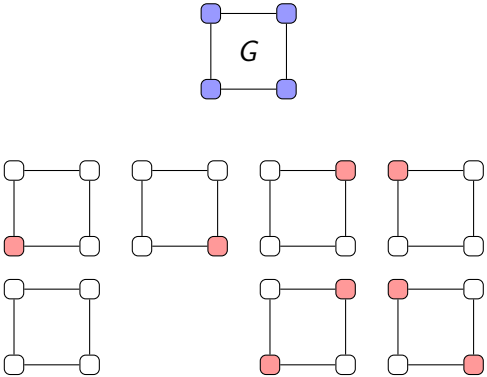

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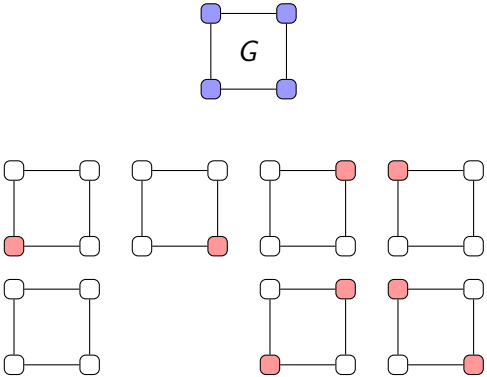

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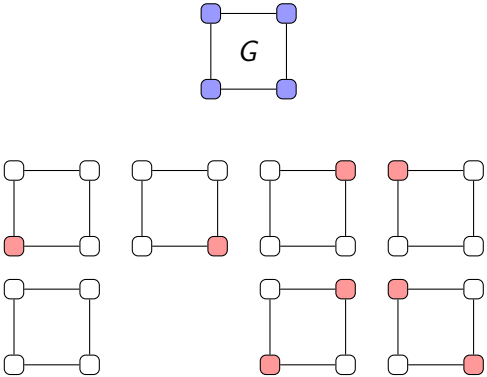
- Through *partition function estimation*.



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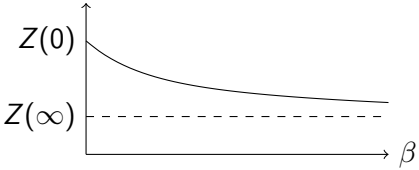

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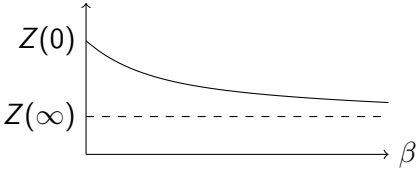

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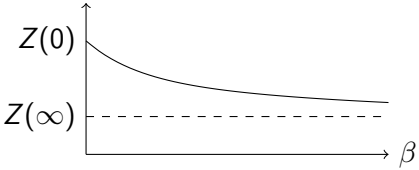

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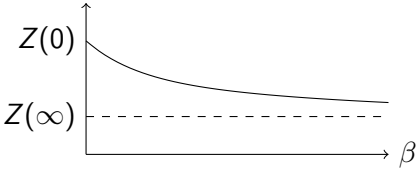

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- Applications:
  - Counting independent sets.
  - Counting k-colorings.
  - Counting matchings.
  - Computing the volume of a convex body.

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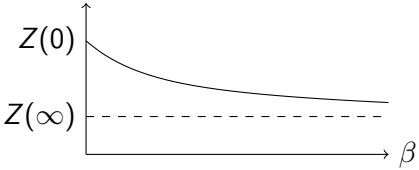

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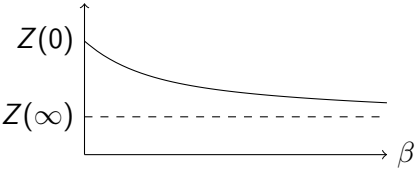

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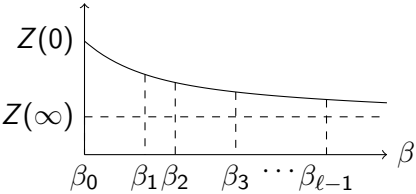

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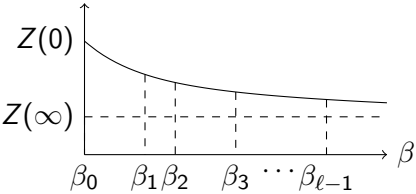

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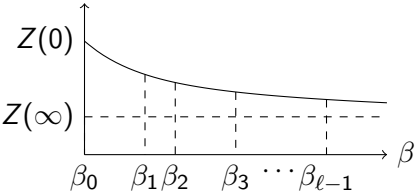

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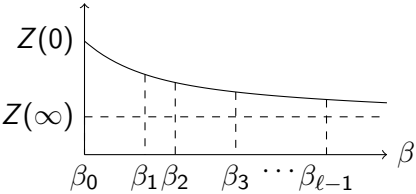

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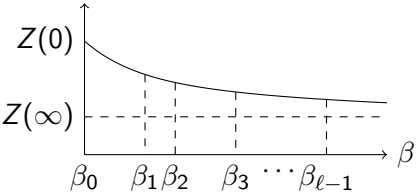

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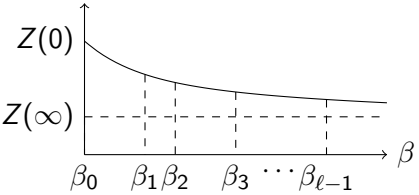

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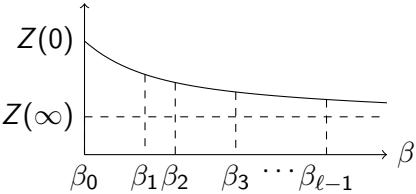

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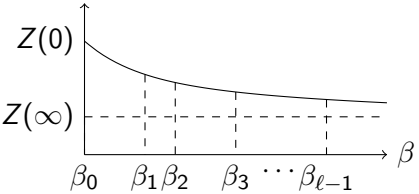

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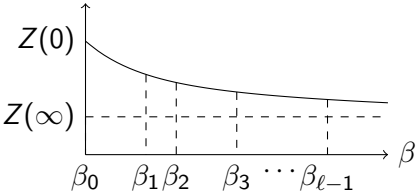
Steps required: $\mathcal{O}(\ell \cdot \frac{\ell}{\varepsilon^2} \cdot \mathrm{MT})$.



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Applications:

	Classical	Quantum, prev.	Quantum, new
Independent set	$\widetilde{\mathcal{O}}(\frac{\lvert V\rvert^2}{\varepsilon^2})$	$\widetilde{\mathcal{O}}(\frac{\lvert V\rvert^{1.5}}{\varepsilon})$	$\widetilde{\mathcal{O}}(\frac{\lvert V\rvert^{1.25}}{\varepsilon})$
Graph colorings	$\widetilde{\mathcal{O}}(\frac{\lvert V\rvert^2}{\varepsilon^2})$	$\widetilde{\mathcal{O}}(\frac{\lvert V\rvert^{1.5}}{\varepsilon})$	$\widetilde{\mathcal{O}}(\frac{\lvert V\rvert^{1.25}}{\varepsilon})$
Graph matchings	$\widetilde{\mathcal{O}}(\frac{\lvert V\rvert\lvert E\rvert}{\varepsilon^2})$	$\widetilde{\mathcal{O}}(\frac{\lvert V\rvert\lvert E\rvert^{0.5}}{\varepsilon})$	$\widetilde{\mathcal{O}}(\frac{\lvert V\rvert^{0.75}\lvert E\rvert^{0.5}}{\varepsilon})$
Volume convex body	$\widetilde{\mathcal{O}}(d^{3.5} + \frac{d^2}{\varepsilon^2})$	$\widetilde{\mathcal{O}}(d^3 + \frac{d^{2.5}}{\varepsilon})$	$\widetilde{\mathcal{O}}(d^3 + \frac{d^{2.25}}{\varepsilon})$

Results

Long line of work:

[DF91;BŠVV08]	$\widetilde{\mathcal{O}}(\frac{\log^2 \lvert\Omega\rvert}{\varepsilon^2} \cdot \mathrm{MT})$
[ŠVV09; Hub15; Kol18]	$\widetilde{\mathcal{O}}(\frac{\log \lvert\Omega\rvert}{\varepsilon^2} \cdot \mathrm{MT})$
[WCNA09]	$\widetilde{\mathcal{O}}(\frac{\log^2 \lvert\Omega\rvert}{\varepsilon} \cdot \sqrt{\mathrm{MT}})$
[Mon15]	$\widetilde{\mathcal{O}}(\log \lvert\Omega\rvert \cdot (\frac{1}{\varepsilon}\sqrt{\mathrm{MT}} + \mathrm{MT}))$
[HW20; AHN+22]	$\widetilde{\mathcal{O}}(\frac{\log \lvert\Omega\rvert}{\varepsilon} \cdot \sqrt{\mathrm{MT}})$
This work	$\widetilde{\mathcal{O}}(\frac{\log^{3/4} \lvert\Omega\rvert}{\varepsilon} \cdot \sqrt{\mathrm{MT}})$

Applications:

	Classical	Quantum, prev.	Quantum, new
Independent set	$\widetilde{\mathcal{O}}(\frac{\lvert V\rvert^2}{\varepsilon^2})$	$\widetilde{\mathcal{O}}(\frac{\lvert V\rvert^{1.5}}{\varepsilon})$	$\widetilde{\mathcal{O}}(\frac{\lvert V\rvert^{1.25}}{\varepsilon})$
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- *Classical cost:* $\mathcal{O}\left(\ell \cdot \frac{\ell}{\varepsilon^2} \cdot \mathrm{MT}\right)$
- *Quantum cost:* $\widetilde{\mathcal{O}}\left(\ell \cdot \sqrt{\frac{\ell}{\varepsilon^2} \cdot \mathrm{MT}}\right)$ (this work)

Unbiased, non-destructive mean estimation

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New component:

Unbiased, non-destructive mean estimation

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Given

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  $O_X : |\omega\rangle |0\rangle \mapsto |\omega\rangle |X(\omega)\rangle$,
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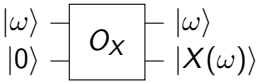
$$\frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2} = \mathcal{O}(1)$$



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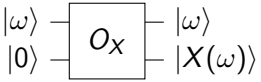

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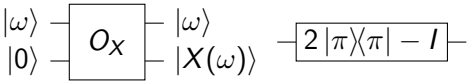

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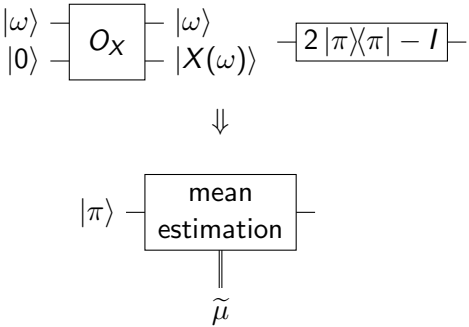

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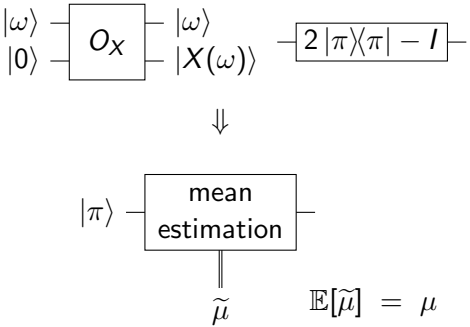

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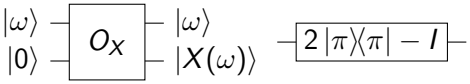
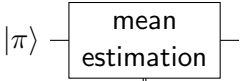


$$\Downarrow$$



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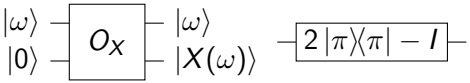

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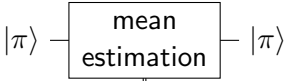

$$\widetilde{\mu}$$

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$$\frac{\mathsf{Var}[\widetilde{\mu}]}{\mathbb{E}[\widetilde{\mu}]^2} \leq \varepsilon^2$$

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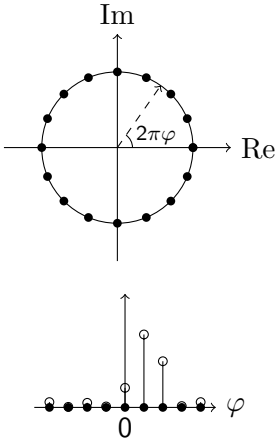

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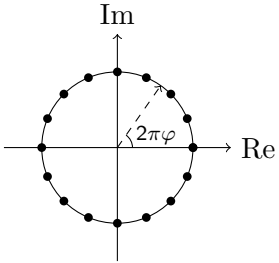

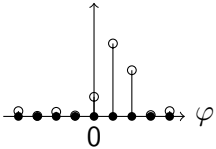

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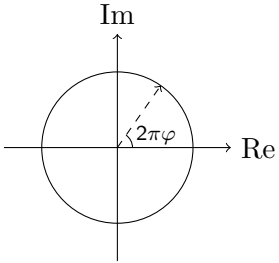

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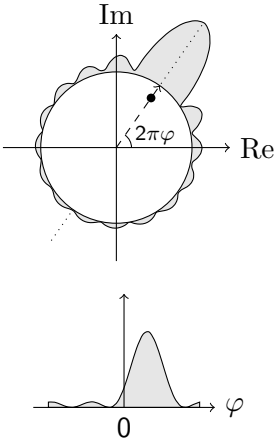

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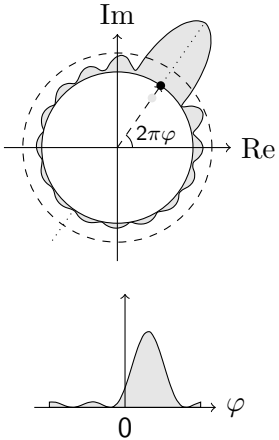

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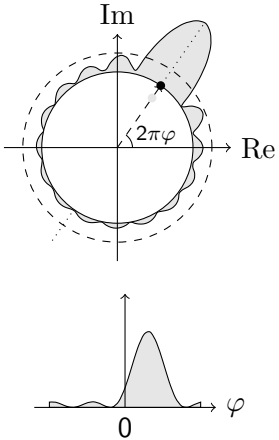

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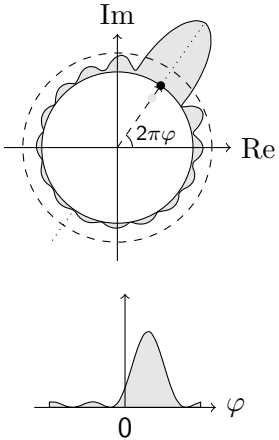
- Plug into mean estimation routines [Mon15].



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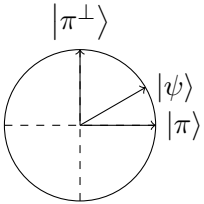

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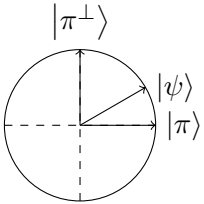

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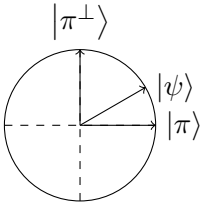

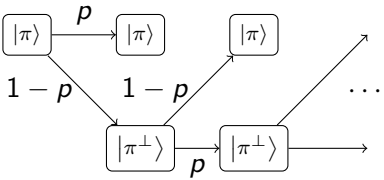

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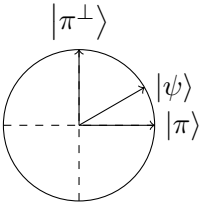

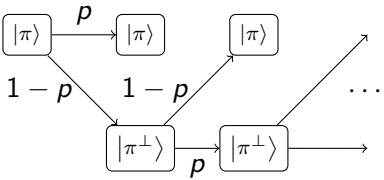

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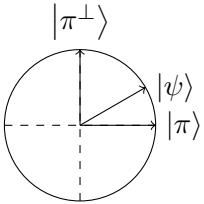

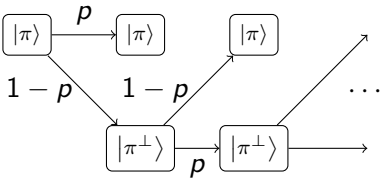

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Thanks for your attention!

arjan@cwi.nl

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