Improved Quantum Query Upper Bounds Based on Classical Decision Trees

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Research Center for Quantum Software

Decision trees

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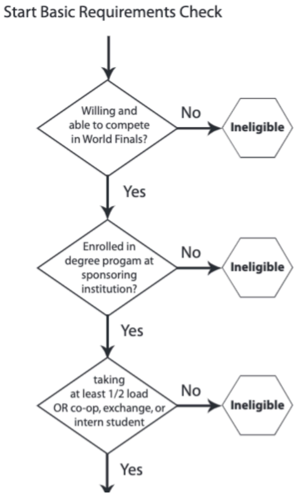
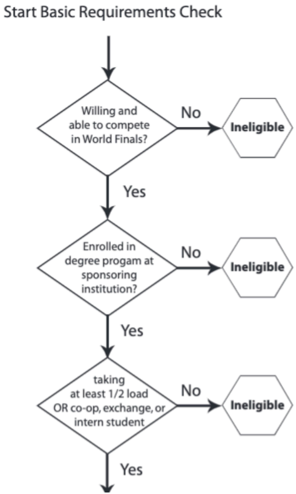


Decision trees

In general:

1. Rooted tree.
2. Every node has a decision rule.
3. Leafs are labeled by outputs.



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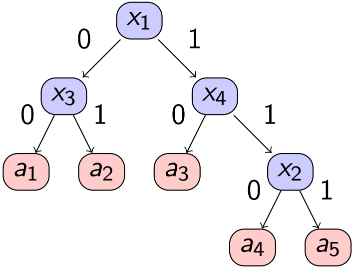
For the purposes of this talk:

1. Input is a bit string $x \in \{0,1\}^n$,
2. Nodes are single bit queries.
3. Decision tree defines $f : \{0,1\}^n \to A$.

$x \in \{0,1\}^n$



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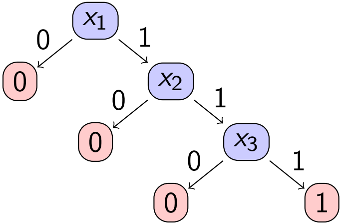
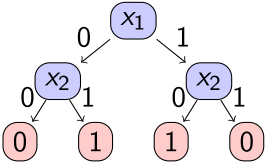
Examples:

1. *AND*-decision tree.
2. *PARITY*-decision tree.

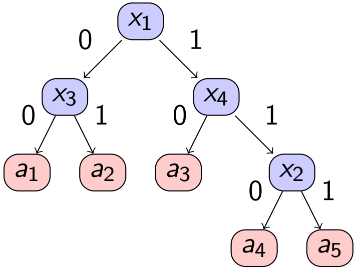
AND-tree



PARITY-tree



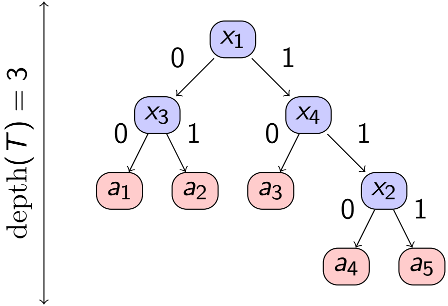
Decision tree measures



Decision tree measures

Measures:

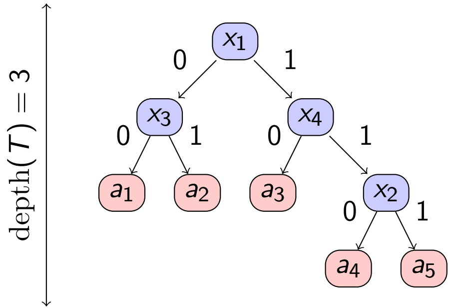
1. $\mathrm{depth}(T)$ – Depth
   Number of layers of decision nodes.



Decision tree measures

Measures:

1. $\mathrm{depth}(T)$ – Depth
   Number of layers of decision nodes.
2. $\mathrm{size}(T)$ – Size
   Number of decision nodes.

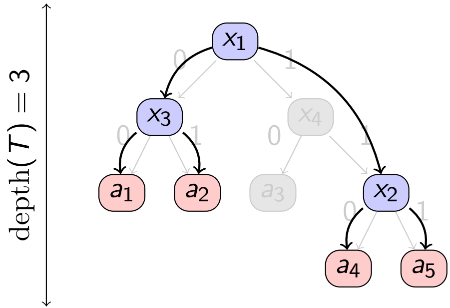


$\mathrm{size}(T) = 4$

Decision tree measures

Measures:

1. $\mathrm{depth}(T)$ – Depth
   Number of layers of decision nodes.
2. $\mathrm{size}(T)$ – Size
   Number of decision nodes.
3. $\mathrm{rank}(T)$ – Rank
   Depth of largest full binary subtree.



$$\begin{aligned}\mathrm{size}(T) &= 4\\ \mathrm{rank}(T) &= 2\end{aligned}$$

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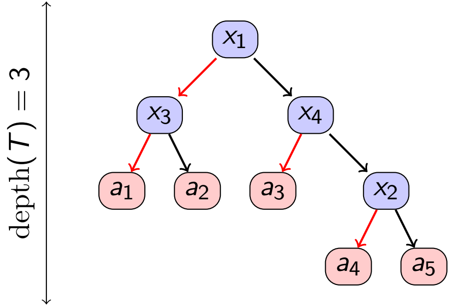
   Depth of largest full binary subtree.
4. $G(T)$ – Guessing complexity
   Most number of red edges in path.



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Function measures

We can lift decision tree measures to function measures.

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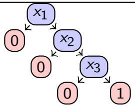
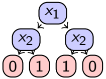
T		
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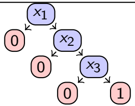
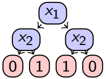
T		
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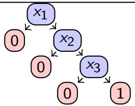
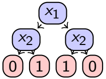
T		
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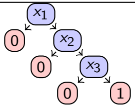
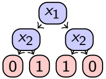
T		
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3. Can make a big difference!
   - $\exists f : \mathrm{rdepth}(f) \ll \mathrm{depth}(f)$ [SW86;ABB+17;MRS18]

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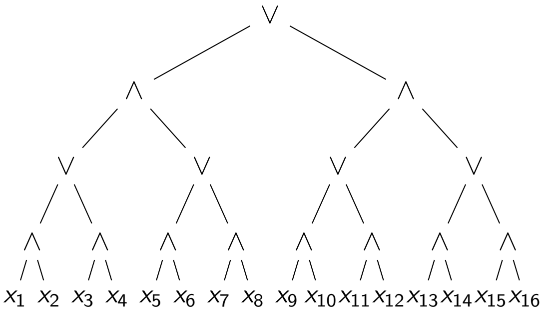

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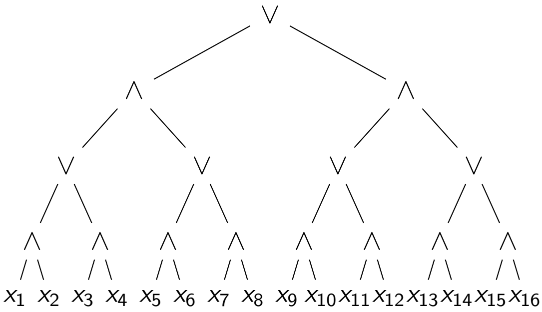

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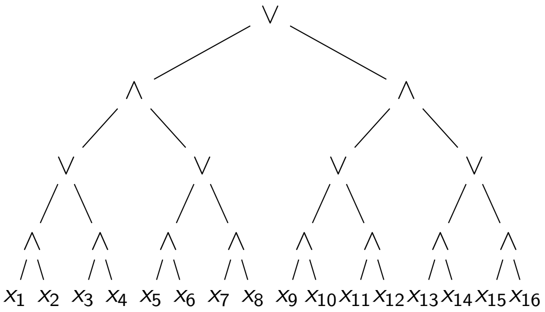

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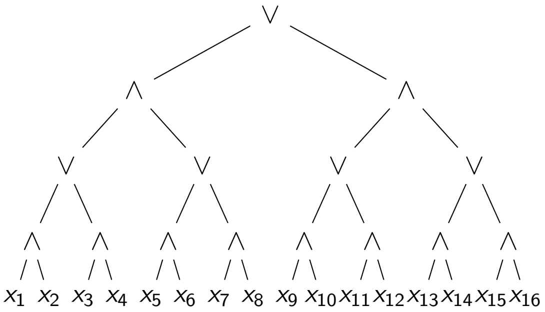

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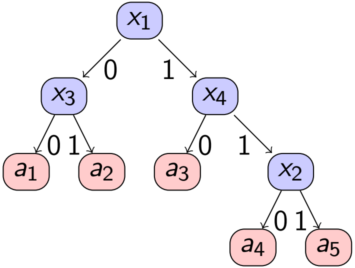

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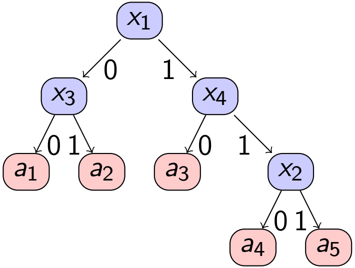

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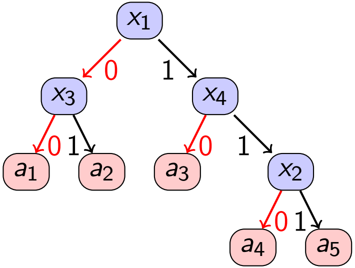

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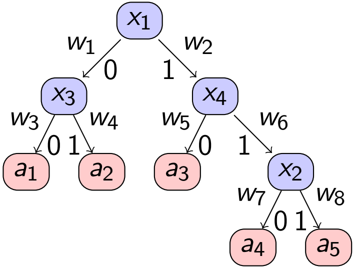

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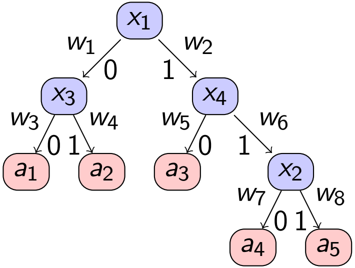

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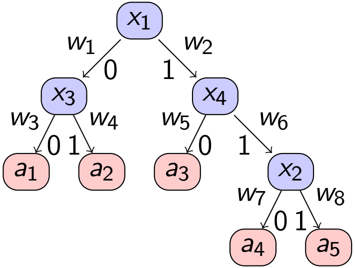

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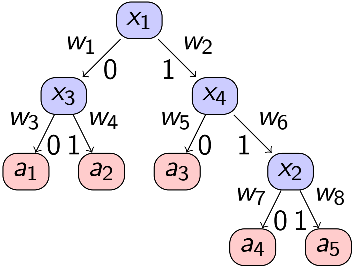

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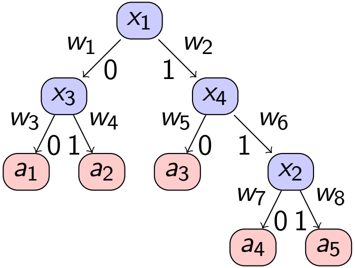

Our contribution: we provide the optimal weight assignment.

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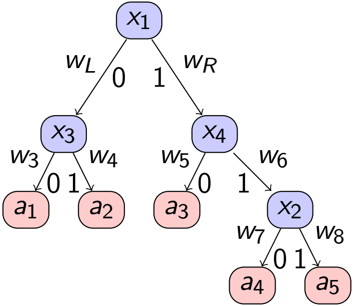

Decision tree to quantum algorithm – optimal weight assignment

Goal: Take a decision tree and construct a quantum algorithm from it.

1. Construction of [BT20]: Let
   - $W_+ = \max_P \{\sum_{e \in P} \frac{1}{w_e}\}$.
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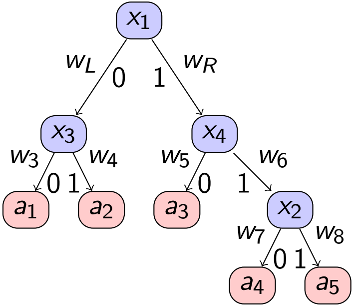
   - $C = \sqrt{W_+ W_-}$.

   $\Rightarrow \mathcal{O}(C)$-query algorithm.



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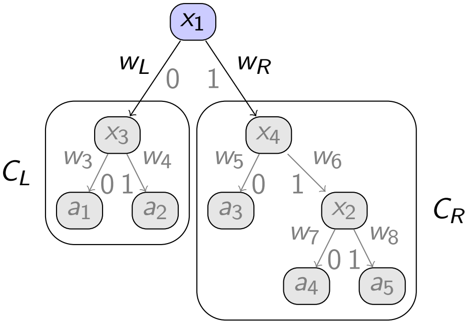

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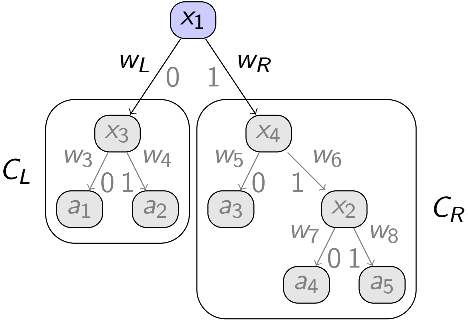

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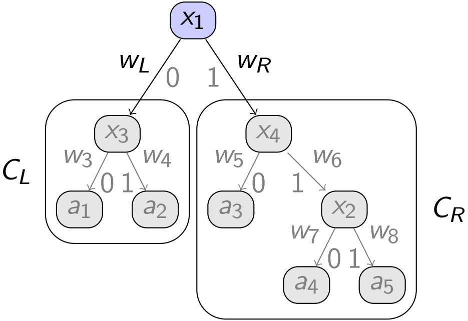

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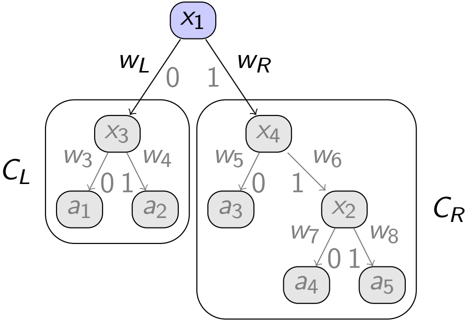

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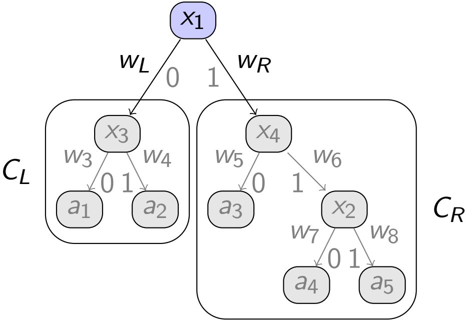
3. $\Rightarrow \mathcal{O}(\sqrt{\mathrm{size}(T)})$-query algorithm.
4. $\exists T : \sqrt{\mathrm{size}(T)} \ll \sqrt{G(T)\mathrm{depth}(T)}$.



$$w_R = \frac{1}{w_L} = \frac{C_L - C_R + \sqrt{(C_L - C_R)^2 + 4}}{2}.$$

$$\Rightarrow C = \frac{C_L + C_R + \sqrt{(C_L - C_R)^2 + 4}}{2}.$$

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Thanks for your attention!

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