Quantum algorithms through graph composition

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Quantum algorithmic frameworks

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Setting:

1. $f : \mathcal{D} \to \{0,1\}$.
2. $\mathcal{D} \subseteq \{0,1\}^n$.
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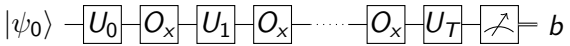

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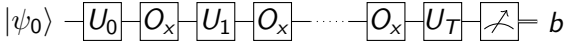

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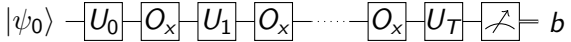

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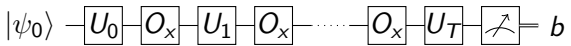

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[Rei11]

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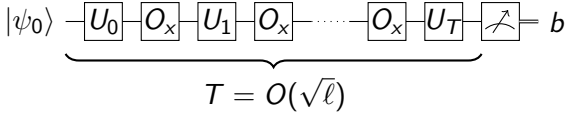

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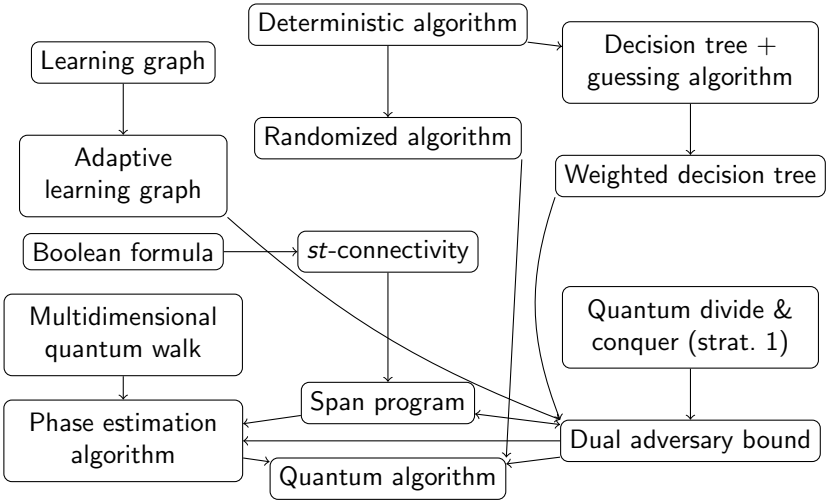

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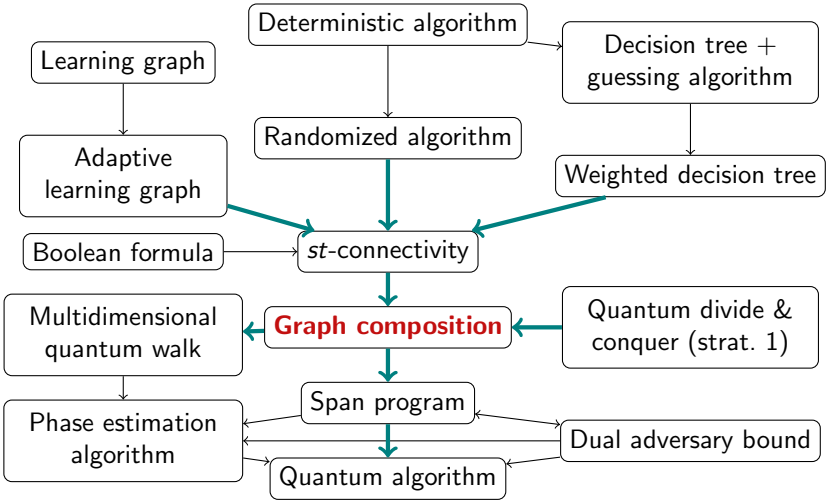

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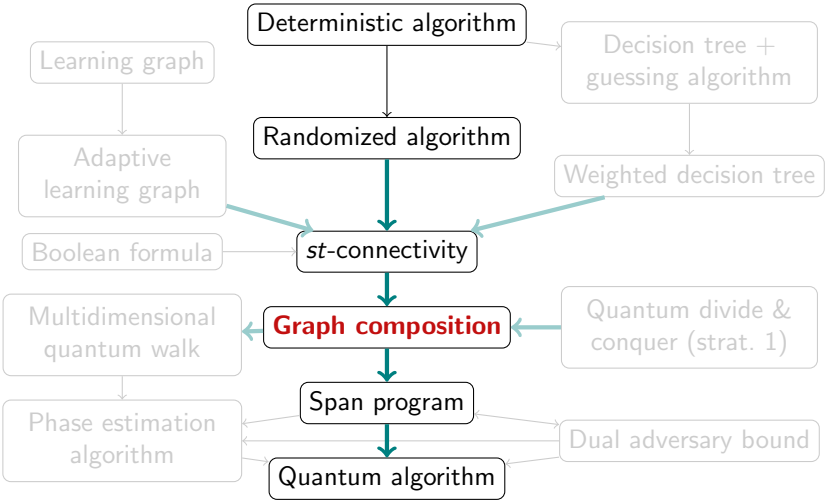
This talk:

1. Span programs
2. Graph composition
3. *st*-connectivity examples
4. Randomized → *st*-connectivity



Span programs [RŠ12,Rei09,Rei11]

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Four ingredients:

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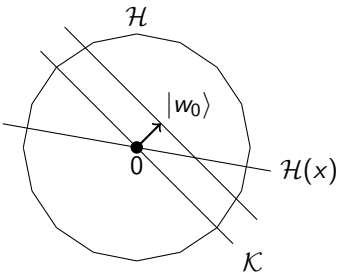

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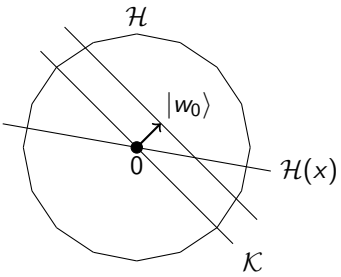

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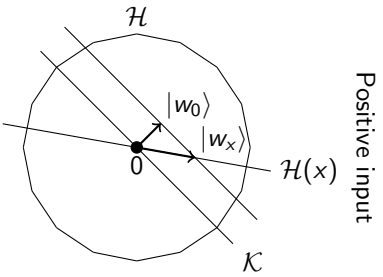

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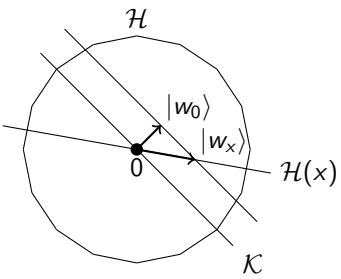

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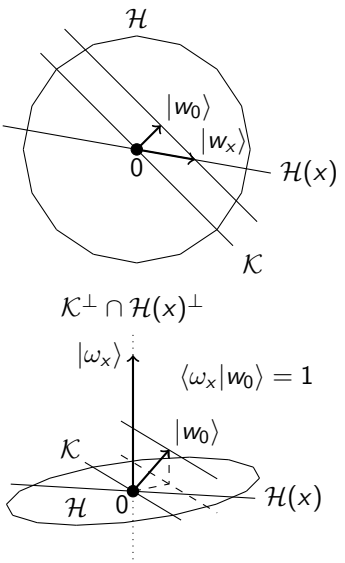

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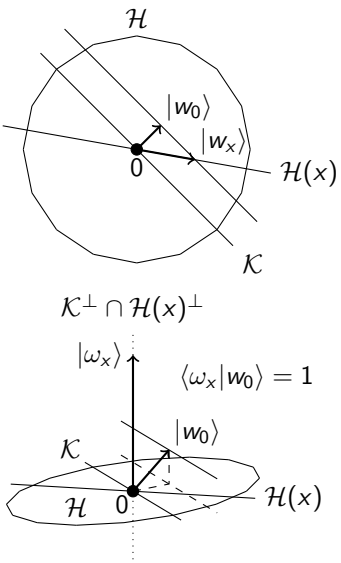
Thm: "Complete" for computing boolean functions.



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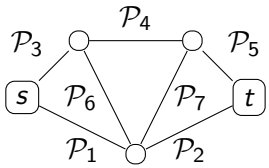

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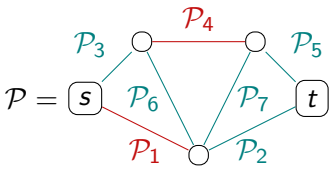

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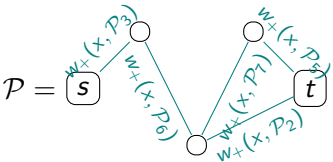

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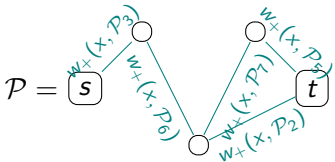

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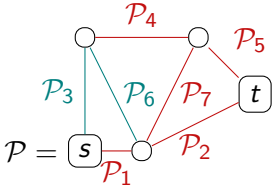

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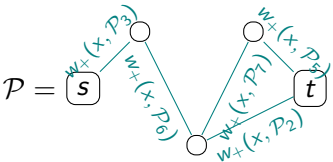

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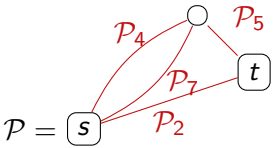

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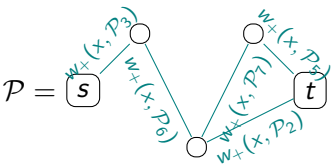

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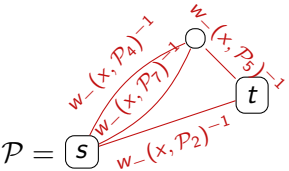

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Path-cut theorem:

1. For P an st-path: $w_+(x, \mathcal{P}) \leq \sum_{e \in P} w_+(x, \mathcal{P}_e)$.
2. For C an st-cut: $w_-(x, \mathcal{P}) \leq \sum_{e \in C} w_-(x, \mathcal{P}_e)$.

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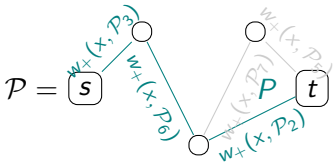

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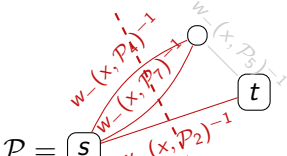

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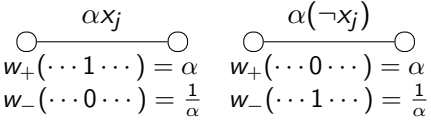

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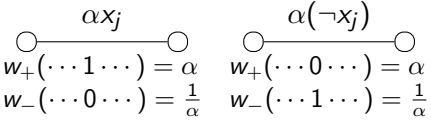

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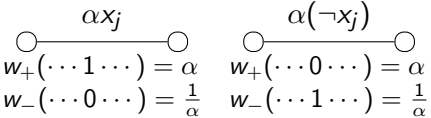
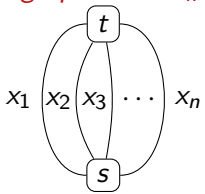


st-graph for OR_n*:*



st-connectivity [BR12,JK17,JJKP18]

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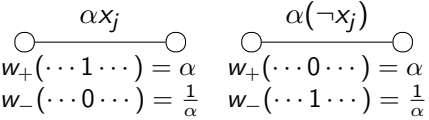

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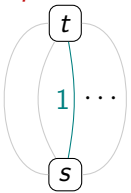

$x = 0010 \cdots 0 \Rightarrow w_+(x) \leq 1$

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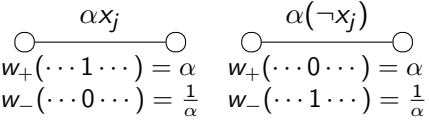

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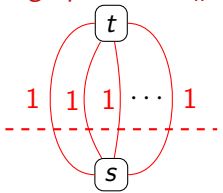

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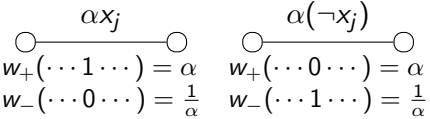

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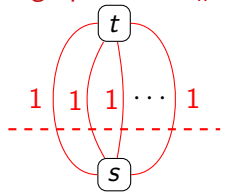

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Quadratic speed-up for search.

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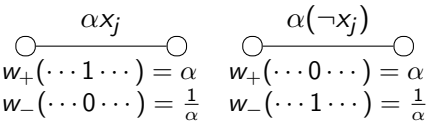

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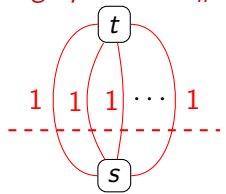

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Example: Threshold function [This work]

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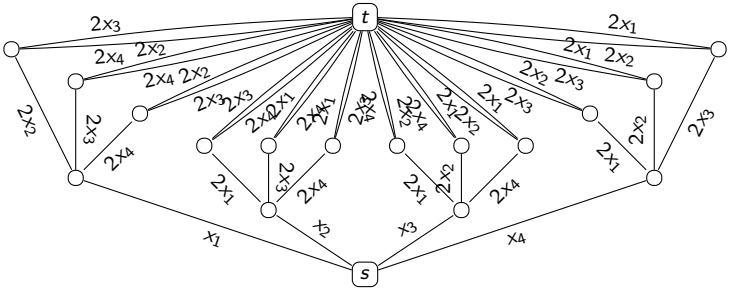

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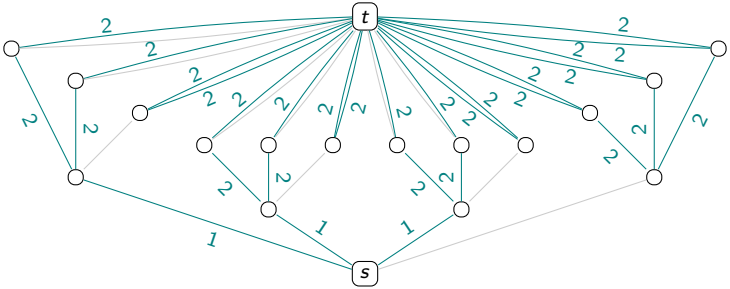

$x = 1110 \Rightarrow w_+(x) = 1$

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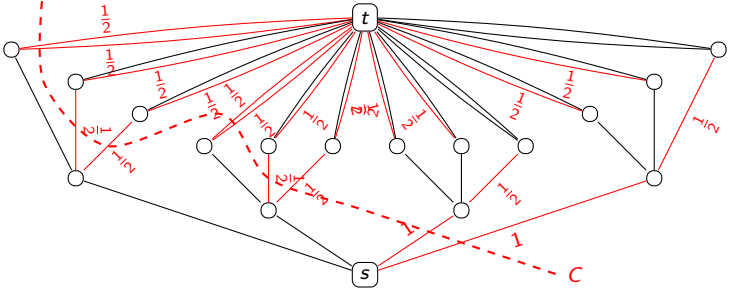

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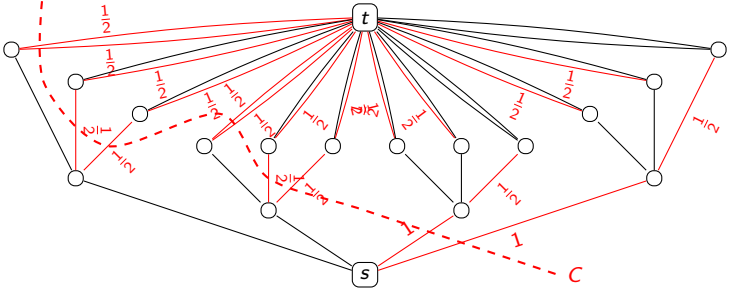

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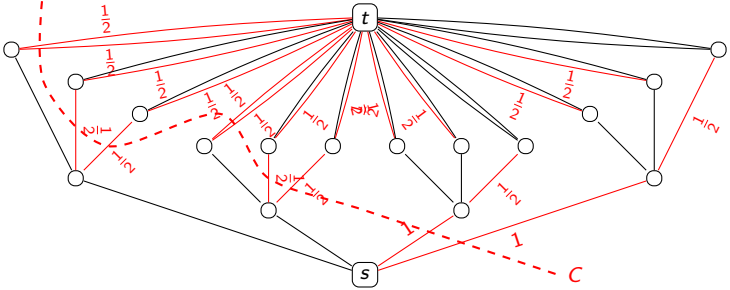

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Remark: With $k = n/2$, it also solves gapped majority in $O(1)$ queries.

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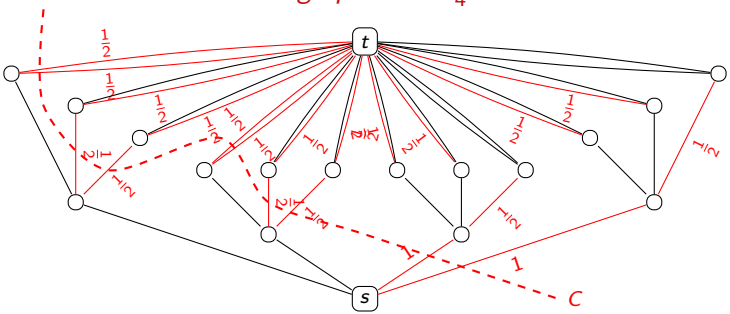

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Classical algorithms $\to$ st-connectivity [This work]

Classical algorithms $\to$ *st*-connectivity [This work]

Deterministic $\to$ st-connectivity:

Classical algorithms $\to$ *st*-connectivity [This work]

Deterministic $\to$ st-connectivity:

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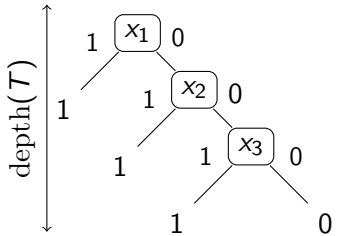

Classical algorithms $\to$ *st*-connectivity [This work]

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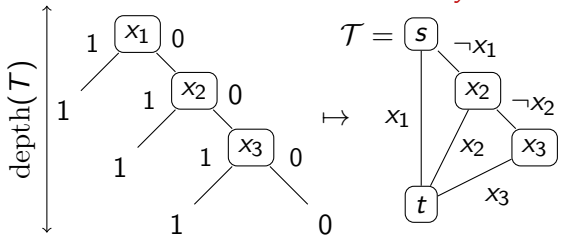

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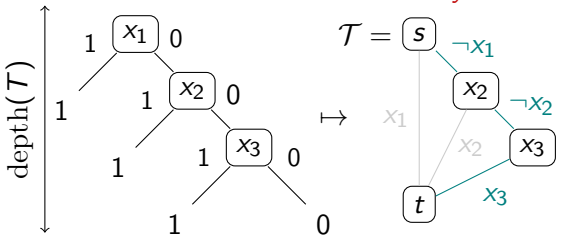

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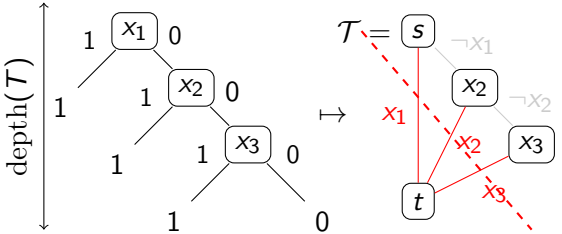

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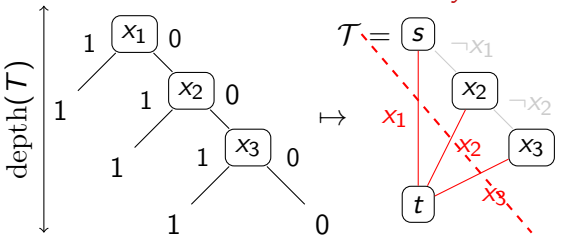

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Randomized $\to$ st-connectivity: (sketch)

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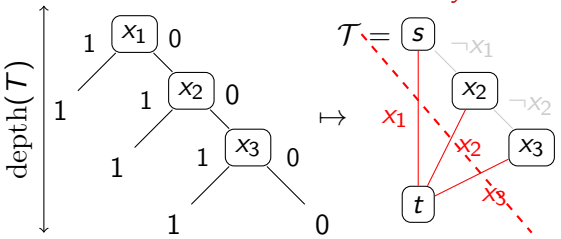

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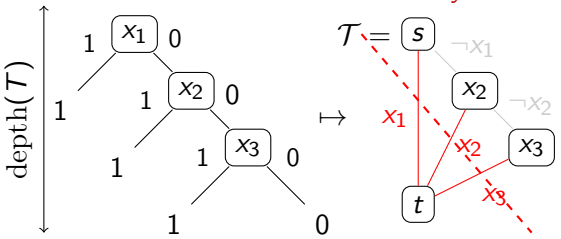

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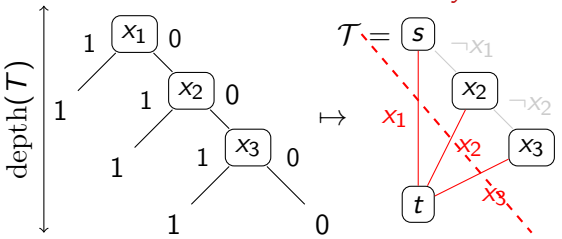

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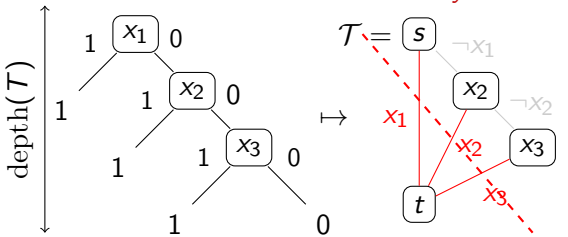
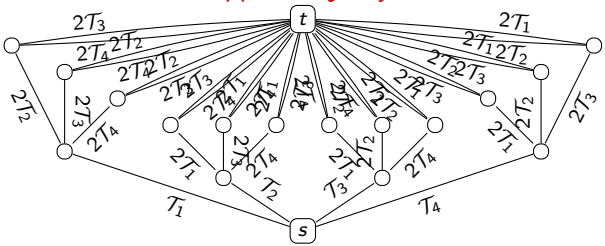


Gapped majority:



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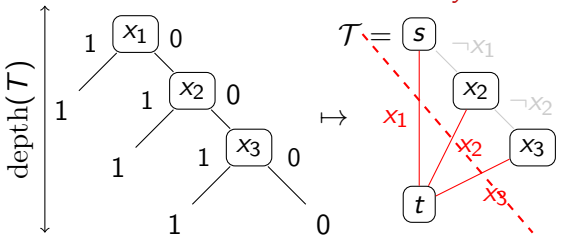

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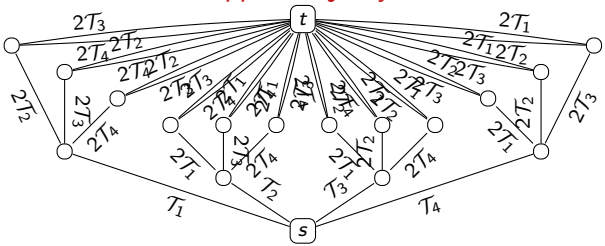

Summary

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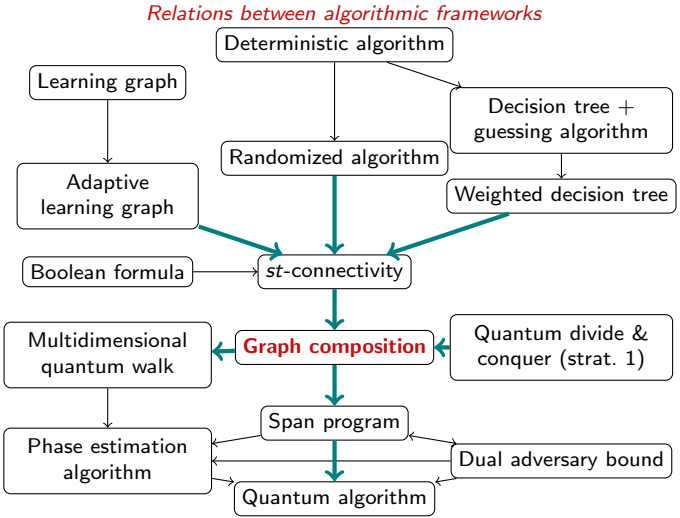
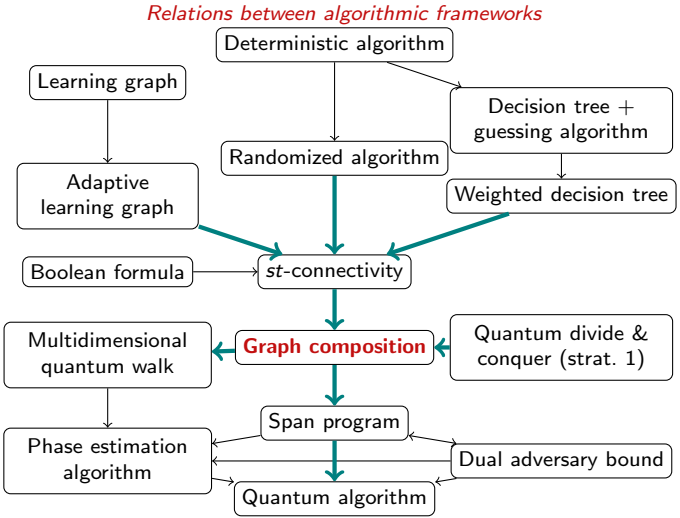

Relations between algorithmic frameworks
Deterministic algorithm
Learning graph
Decision tree + guessing algorithm
Randomized algorithm
Adaptive learning graph
Weighted decision tree
Boolean formula
st-connectivity
Multidimensional quantum walk
Graph composition
Quantum divide & conquer (strat. 1)
Span program
Phase estimation algorithm
Dual adversary bound
Quantum algorithm

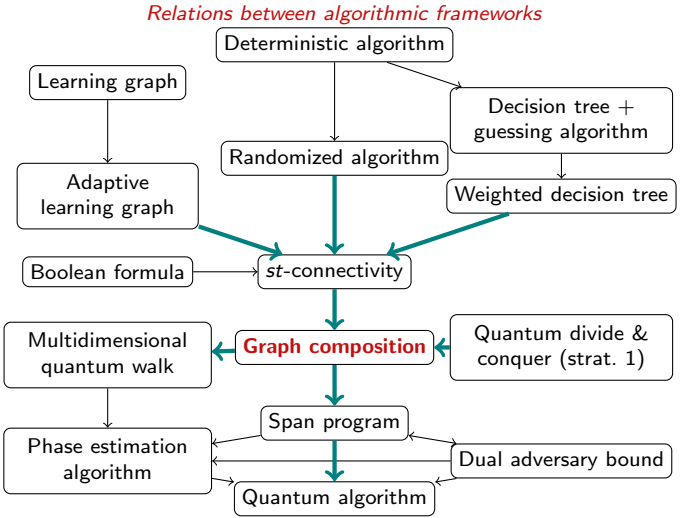
Summary



In this talk:

1. Span programs
2. Graph composition
3. Examples
4. Randomized $\to$ *st*-connectivity.

Summary



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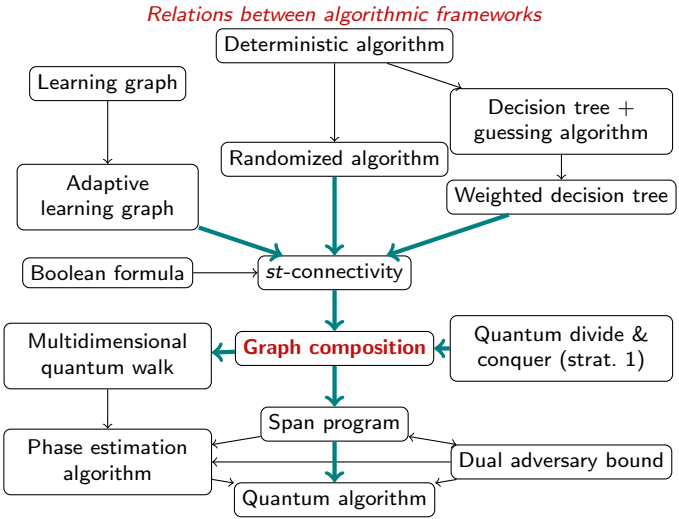
In the papers:

1. Time-efficient implementation of graph composition
2. Generalization to switches
3. Quantum walks frameworks
4. More examples

Summary



In this talk:

1. Span programs
2. Graph composition
3. Examples
4. Randomized $\to$ *st*-connectivity.

In the papers:

1. Time-efficient implementation of graph composition
2. Generalization to switches
3. Quantum walks frameworks
4. More examples

Open question:

Limitations of *st*-connectivity?

Thanks!

Thanks for your attention!

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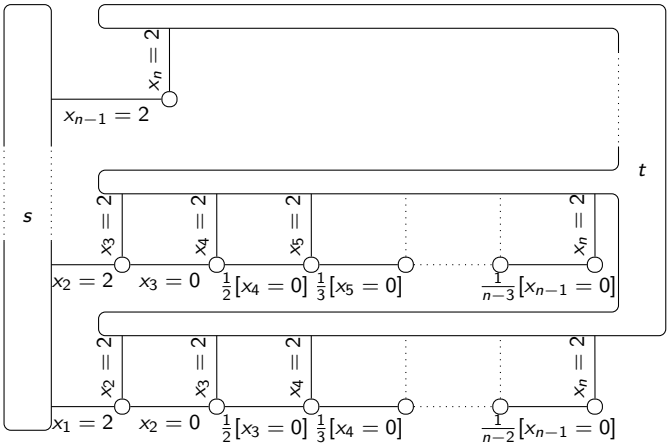

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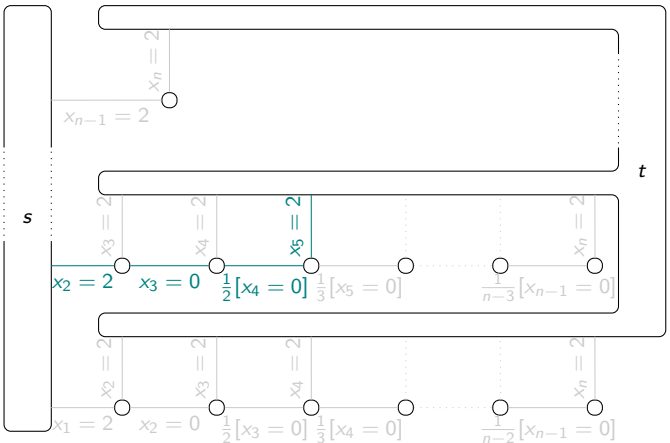

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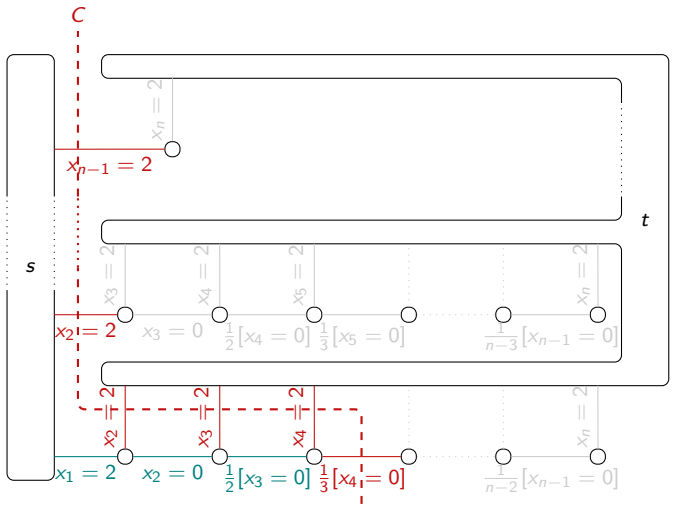

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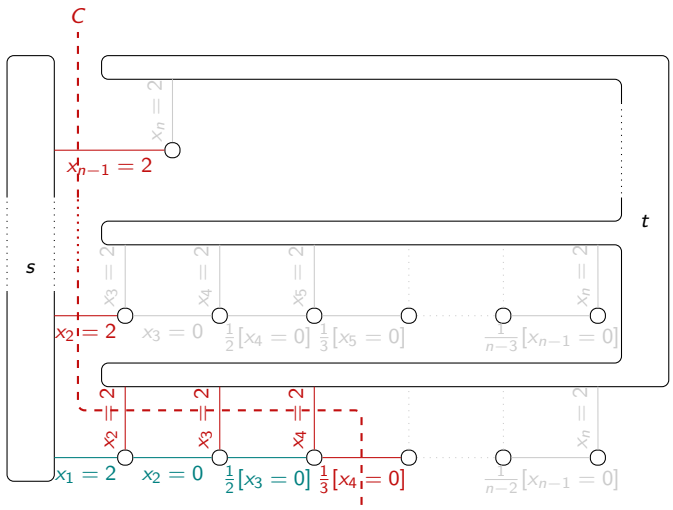
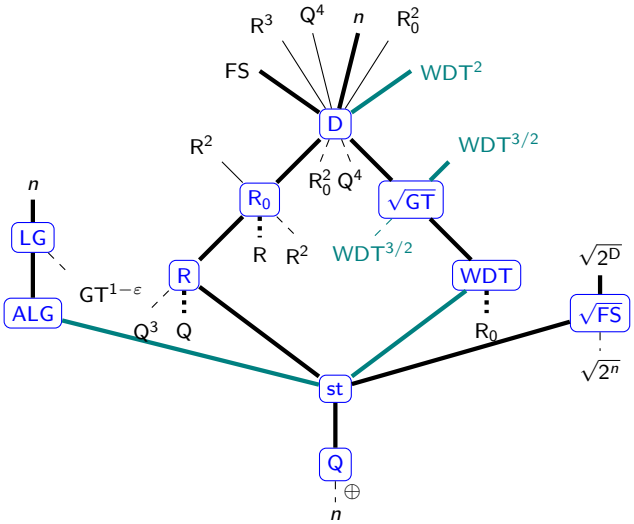
4. $C(\mathcal{P}) \in O(\sqrt{n\log(n)})$.



Complexity measure relations for total boolean functions



Legend:

1. Solid: relation.
2. Dashed: separation.
3. **Fat:** for partial functions.
4. Dotted: unbounded separation.
5. Green: new in this work.

Open questions:

1. Unbounded separation:
   1. Between Q and st?
   2. Between st and R?