Amortized quantum walks with transducers

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September 15th, 2026

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Volume estimation

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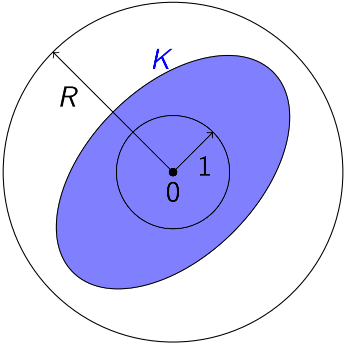

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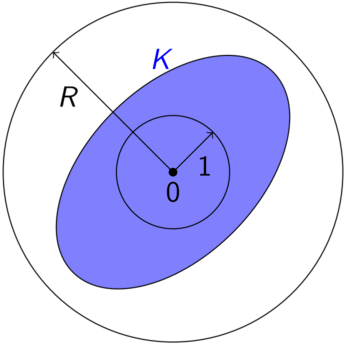

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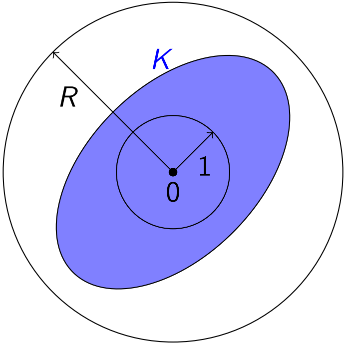

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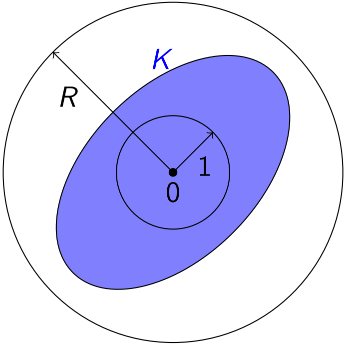

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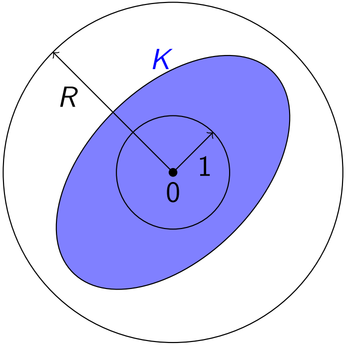

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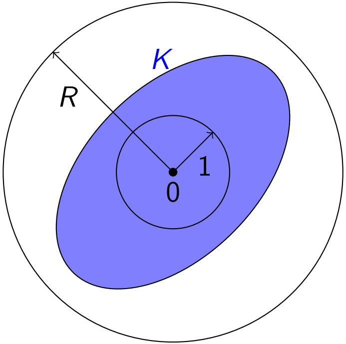

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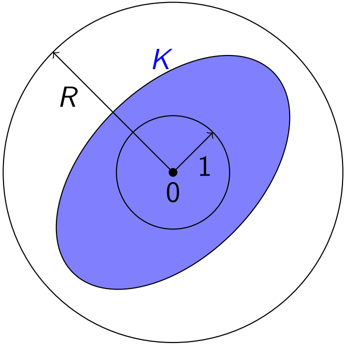

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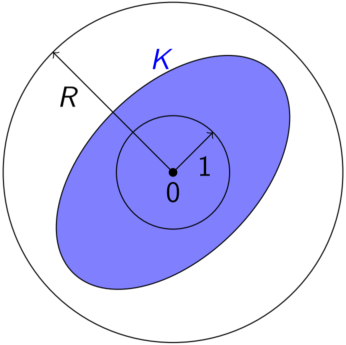

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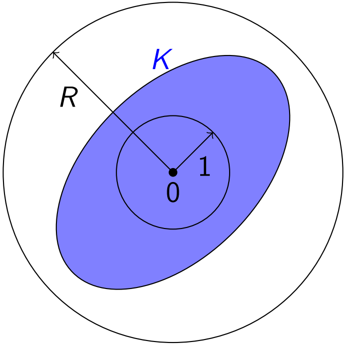
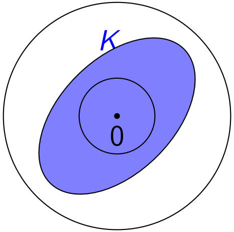


Volume estimation algorithm overview [CV18]



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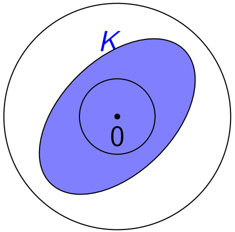

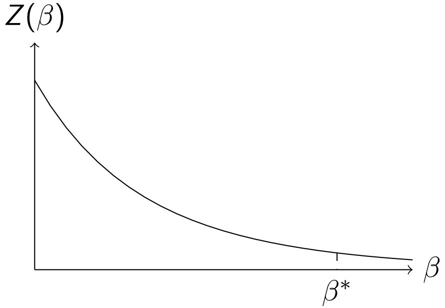

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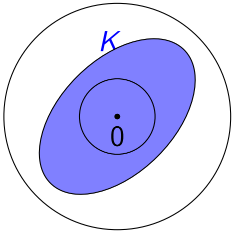

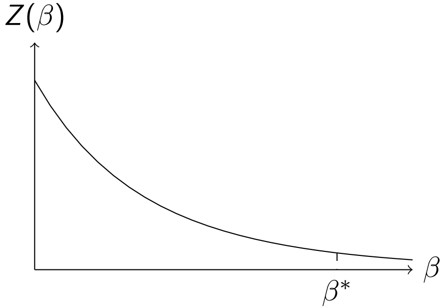

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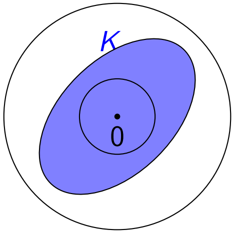

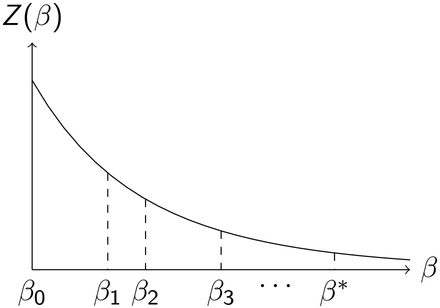

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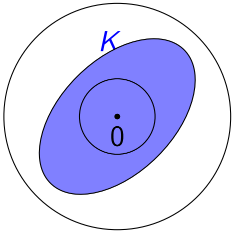

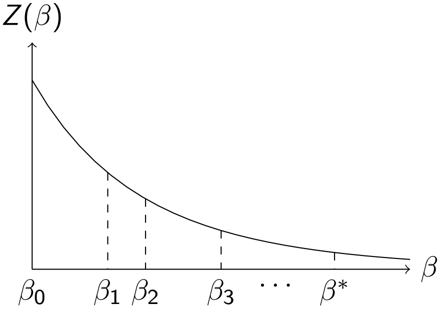

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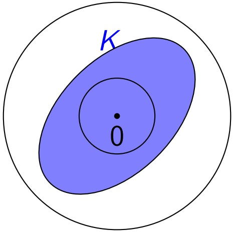

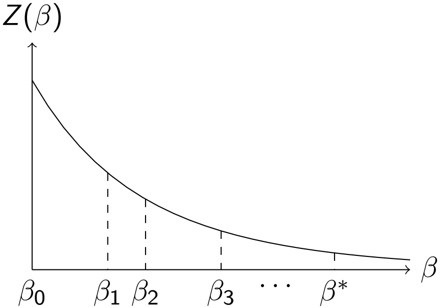

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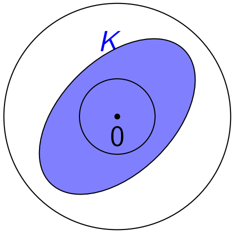

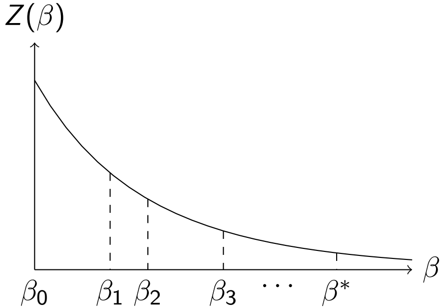

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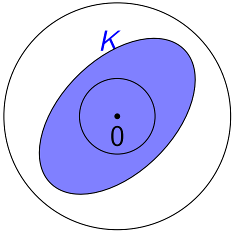
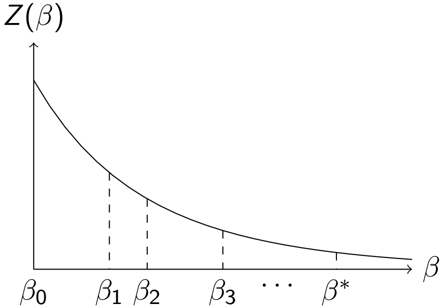
6. *Quantization:* “Use quantum walk”





Random walks

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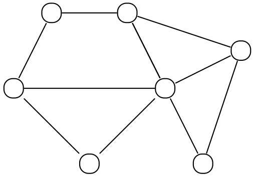
1. *Random walk:*
   1. Connected undirected graph $G = (V, E)$
   2. Weights $w : E \to \mathbb{R}_{>0}$.
      Total weight $\sum_{e \in E} w_e = 1/2$.
   3. Dynamics: $P_{vw} = \frac{w_{vw}}{\sum_{w' \in V} w_{vw'}}$.

$G =$ 

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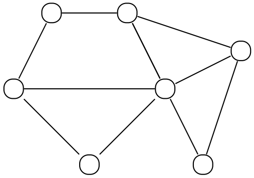
2. *Properties:*
   1. $\sigma(P) \subseteq [-1, 1]$
   2. Unique stationary distribution $\pi P = \pi$
      with $\pi_v = \sum_{w' \in V} w_{vw'} > 0$.
   3. Reversible, i.e., $\pi_v P_{vw} = w_{vw} = \pi_w P_{wv}$.
      (*detailed balance*)

$G =$ 

$$P = \begin{array}{c} v \\ \\ \\ \end{array} \begin{bmatrix} \text{——} \; P_{vw} \; \text{—} \\ \\ \\ \end{bmatrix}$$

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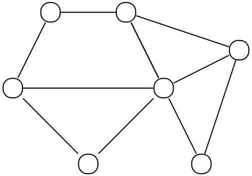
$G =$ 

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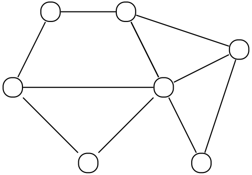
$G =$ 

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   1. Sample from initial distribution $x_0 \sim p$.
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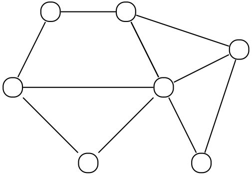
$G =$ 

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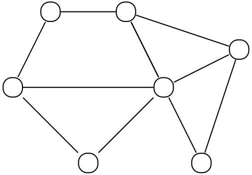
   $G =$ 

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4. *Mixing:* $\chi^2(pP^t \| \pi) \leq (1-\gamma)^{2t}\chi^2(p\|\pi)$.

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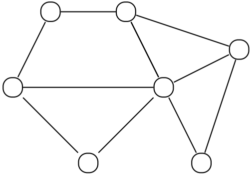
$G =$ 

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4. *Mixing:* $\chi^2(pP^t\|\pi) \leq (1-\gamma)^{2t}\chi^2(p\|\pi)$.
5. *No. steps:* $t \in \Theta\left(\frac{\log\left(\frac{\chi^2(p\|\pi)}{\varepsilon}\right)}{\gamma}\right) \subseteq \widetilde{O}\left(\frac{1}{\gamma}\right)$
   $\Rightarrow \chi^2(pP^t\|\pi) \leq \varepsilon$.

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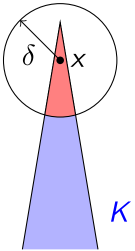

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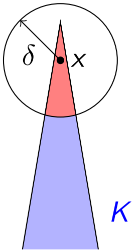

Amortized random walk [CV18]

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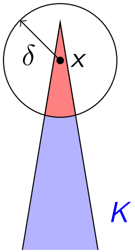

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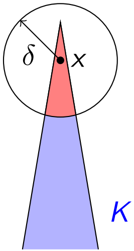

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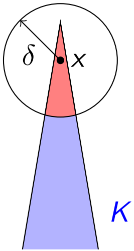

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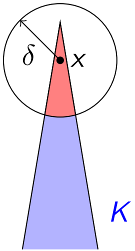
5. *Obstacle:* Quantization?



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Notation:

1. $|*_v\rangle = \sum_{w \in V} \sqrt{P_{vw}} |w\rangle \in \mathbb{C}^V$.
2. $\mathcal{N} : |v\rangle |0\rangle \mapsto |v\rangle |*_v\rangle \in (\mathbb{C}^V)^{\otimes 2}$.
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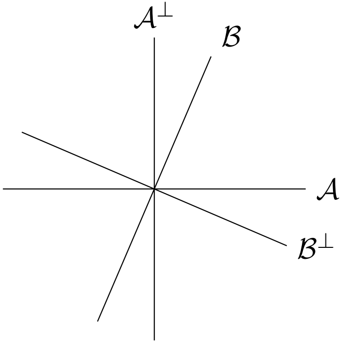

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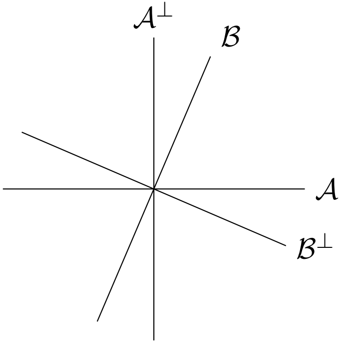

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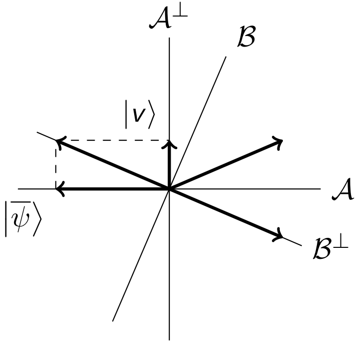

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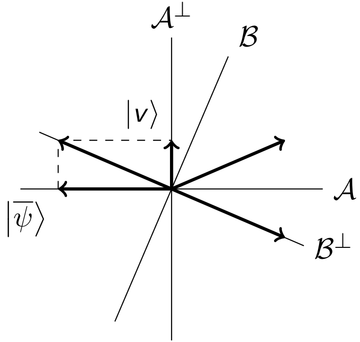
   2. $\mathcal{N}\ket{\overline{\pi}} \overset{U}{\rightsquigarrow} \mathcal{N}\ket{\overline{\pi}}$, whenever $\braket{\overline{\pi}|\pi} = 0$.



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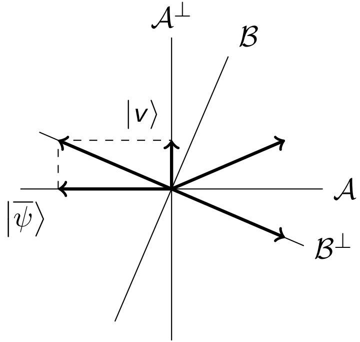
4. *Question:* What is this catalyst state?



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Electrical networks

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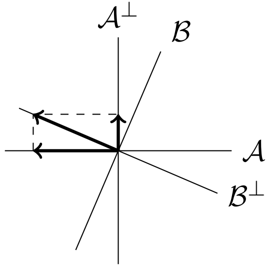

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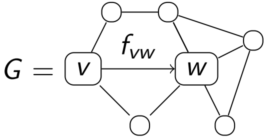

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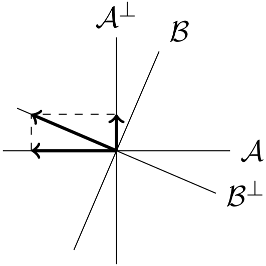

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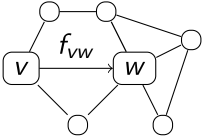
$G =$ 

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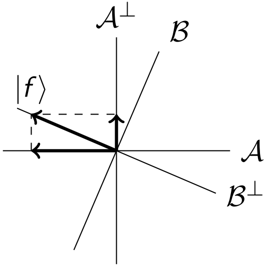

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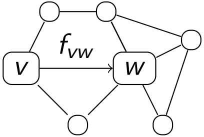
$G =$ 

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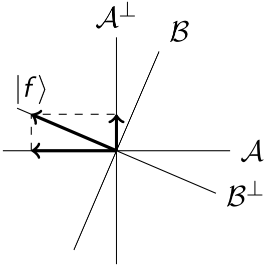
      $\Rightarrow \sum_{v \in V} (\delta_f)_v = 0$.



$$\mathcal{A} = \mathrm{Span}\{|v\rangle |*_v\rangle : v \in V\}$$
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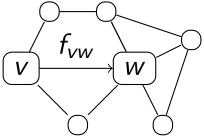
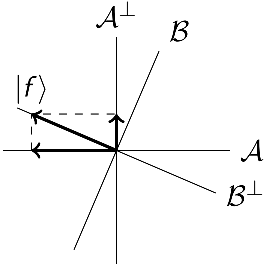
$G =$ 

1. *Properties:*
   1. *Resistance:* $r : E \to \mathbb{R}_{>0}$, $r_{vw} = 1/w_{vw}$.
      $\Rightarrow |*_v\rangle = \sum_{w \in V} \frac{1}{\sqrt{\pi_v r_{vw}}} |w\rangle$
   2. *Flows:* $|f\rangle = \sum_{vw \in E} f_{vw} \sqrt{r_{vw}} |vw\rangle \in \mathcal{B}^\perp$
      $\Rightarrow f_{vw} = -f_{wv}$.
   3. *Net-flow:* $(\delta_f)_v = \sum_w f_{vw}$.
      $\Rightarrow \sum_{v \in V} (\delta_f)_v = 0$.
   4. *Effective resistance:*
      $R_{\mathrm{eff}}(G, r; \overline{\pi}) = \min_{f : \delta_f = \overline{\pi}} \| |f\rangle \|^2$.



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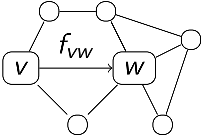
$G =$ 

Electrical networks

1. *Properties:*
   1. *Resistance:* $r : E \to \mathbb{R}_{>0}$, $r_{vw} = 1/w_{vw}$.
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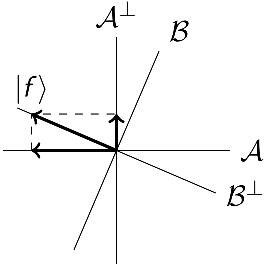
   4. *Effective resistance:*
      $R_{\mathrm{eff}}(G, r; \overline{\pi}) = \min_{f : \delta_f = \overline{\pi}} \||f\rangle\|^2$.
2. *Lemma:* $\Pi_{\mathcal{A}} |f\rangle = \mathcal{N} \sum_{v \in V} \frac{(\delta_f)_v}{\sqrt{\pi_v}} |v\rangle$.



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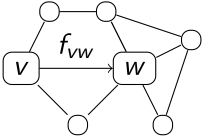
$G =$ 

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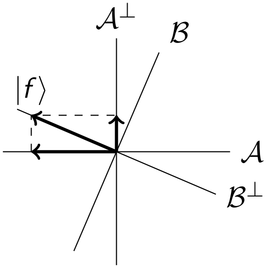
3. $\langle\overline{\pi}|\pi\rangle = 0 \Rightarrow |\overline{\pi}\rangle = \sum_{v \in V} \frac{\overline{\pi}_v}{\sqrt{\pi_v}} |v\rangle$,
   where $\overline{\pi}$ is a net-flow.



$$\mathcal{A} = \text{Span}\{|v\rangle\,|*_v\rangle : v \in V\}$$
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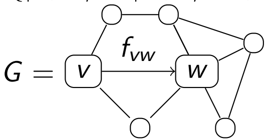

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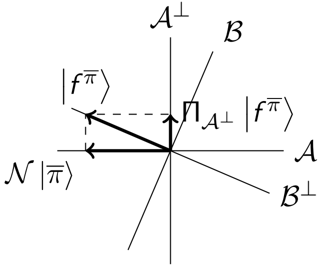
3. $\langle \overline{\pi} | \pi \rangle = 0 \Rightarrow |\overline{\pi}\rangle = \sum_{v \in V} \frac{\overline{\pi}_v}{\sqrt{\pi_v}} |v\rangle$,
   where $\overline{\pi}$ is a net-flow.
4. *Conclusion:* $\mathcal{N} |\overline{\pi}\rangle \overset{U}{\rightsquigarrow} -\mathcal{N} |\overline{\pi}\rangle$ with
   complexity $R_{\mathrm{eff}}(G, r; \overline{\pi}) - 1$.



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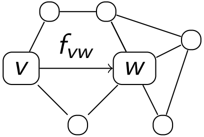
$G =$ 

Input-invariant reflection transducer

Input-invariant reflection transducer

1. *Theorem:* U acts as
   1. $\mathcal{N}\,|\pi\rangle \overset{U}{\rightsquigarrow} \mathcal{N}\,|\pi\rangle$ with complexity 0
   2. $\mathcal{N}\,|\overline{\pi}\rangle \overset{U}{\rightsquigarrow} -\mathcal{N}\,|\overline{\pi}\rangle$ with complexity $R_{\mathrm{eff}}(G, r; \overline{\pi}) - 1$.
   3. Public space: $\mathcal{A} = \mathrm{Span}\{|v\rangle\,|*_v\rangle : v \in V\}$.
   4. Private space: $\mathcal{A}^\perp$.

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2. *Problem:* depend on the input.

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2. *Problem:* depend on the input.
3. *Naive solution:* $-\boxed{\mathcal{N}}-\boxed{U}-\boxed{\mathcal{N}^{\dagger}}-$

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   3. Public space: $\mathcal{A} = \text{Span}\{|v\rangle|*_v\rangle : v \in V\}$.
   4. Private space: $\mathcal{A}^\perp$.
2. *Problem:* depend on the input.
3. *Naive solution:* $-\boxed{\mathcal{N}}-\boxed{U}-\boxed{\mathcal{N}^\dagger}-$
4. $\begin{bmatrix} |\overline{\pi}\rangle \\ \mathcal{N}^\dagger\Pi_{\mathcal{A}^\perp}\left|f^{\overline{\pi}}\right\rangle \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{N}|\overline{\pi}\rangle \\ \Pi_{\mathcal{A}^\perp}\left|f^{\overline{\pi}}\right\rangle \end{bmatrix} \mapsto \cdots \mapsto \begin{bmatrix} -|\overline{\pi}\rangle \\ \mathcal{N}^\dagger\Pi_{\mathcal{A}^\perp}\left|f^{\overline{\pi}}\right\rangle \end{bmatrix}$

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2. *Problem:* depend on the input.
3. *Naive solution:* $\boxed{\mathcal{N}}\!-\!\boxed{U}\!-\!\boxed{\mathcal{N}^\dagger}$
4. $\begin{bmatrix} |\overline{\pi}\rangle \\ \mathcal{N}^\dagger \Pi_{\mathcal{A}^\perp} |f^{\overline{\pi}}\rangle \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{N}|\overline{\pi}\rangle \\ \Pi_{\mathcal{A}^\perp} |f^{\overline{\pi}}\rangle \end{bmatrix} \mapsto \cdots \mapsto \begin{bmatrix} -|\overline{\pi}\rangle \\ \mathcal{N}^\dagger \Pi_{\mathcal{A}^\perp} |f^{\overline{\pi}}\rangle \end{bmatrix}$
5. Feeds everything into $\mathcal{N}$.

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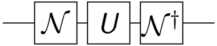
3. *Naive solution:* 

4. $\begin{bmatrix} |\overline{\pi}\rangle \\ \mathcal{N}^\dagger \Pi_{\mathcal{A}^\perp}|f^{\overline{\pi}}\rangle \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{N}|\overline{\pi}\rangle \\ \Pi_{\mathcal{A}^\perp}|f^{\overline{\pi}}\rangle \end{bmatrix} \mapsto \cdots \mapsto \begin{bmatrix} -|\overline{\pi}\rangle \\ \mathcal{N}^\dagger \Pi_{\mathcal{A}^\perp}|f^{\overline{\pi}}\rangle \end{bmatrix}$
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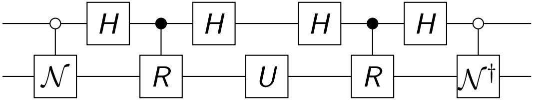
1. *More efficient construction:*



Input-invariant reflection transducer

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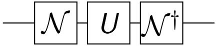
   2. $\mathcal{N}|\overline{\pi}\rangle \overset{U}{\rightsquigarrow} -\mathcal{N}|\overline{\pi}\rangle$ with complexity $R_{\mathrm{eff}}(G, r; \overline{\pi}) - 1$.
   3. Public space: $\mathcal{A} = \mathrm{Span}\{|v\rangle|*_v\rangle : v \in V\}$.
   4. Private space: $\mathcal{A}^\perp$.
2. *Problem:* depend on the input.
3. *Naive solution:*



4. $$\begin{bmatrix} |\overline{\pi}\rangle \\ \mathcal{N}^\dagger \Pi_{\mathcal{A}^\perp} |f^{\overline{\pi}}\rangle \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{N}|\overline{\pi}\rangle \\ \Pi_{\mathcal{A}^\perp} |f^{\overline{\pi}}\rangle \end{bmatrix} \mapsto \cdots \mapsto \begin{bmatrix} -|\overline{\pi}\rangle \\ \mathcal{N}^\dagger \Pi_{\mathcal{A}^\perp} |f^{\overline{\pi}}\rangle \end{bmatrix}$$
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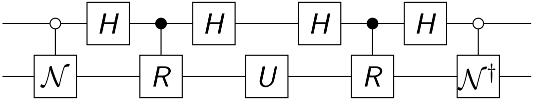

2. $$\begin{bmatrix} |\overline{\pi}\rangle \\ \Pi_{\mathcal{A}^\perp} |f^{\overline{\pi}}\rangle \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{N}|\overline{\pi}\rangle \\ \Pi_{\mathcal{A}^\perp} |f^{\overline{\pi}}\rangle \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{N}|\overline{\pi}\rangle + \Pi_{\mathcal{A}^\perp} |f^{\overline{\pi}}\rangle \\ 0 \end{bmatrix} \mapsto \cdots \mapsto \begin{bmatrix} -|\overline{\pi}\rangle \\ \Pi_{\mathcal{A}^\perp} |f^{\overline{\pi}}\rangle \end{bmatrix}$$

Input-invariant reflection transducer

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   3. Public space: $\mathcal{A} = \text{Span}\{|v\rangle|*_v\rangle : v \in V\}$.
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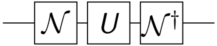
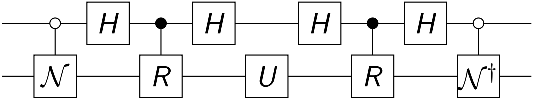
2. *Problem:* depend on the input.
3. *Naive solution:* 
4. $\begin{bmatrix} |\overline{\pi}\rangle \\ \mathcal{N}^\dagger\Pi_{\mathcal{A}^\perp}|f^{\overline{\pi}}\rangle \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{N}|\overline{\pi}\rangle \\ \Pi_{\mathcal{A}^\perp}|f^{\overline{\pi}}\rangle \end{bmatrix} \mapsto \cdots \mapsto \begin{bmatrix} -|\overline{\pi}\rangle \\ \mathcal{N}^\dagger\Pi_{\mathcal{A}^\perp}|f^{\overline{\pi}}\rangle \end{bmatrix}$
5. Feeds everything into $\mathcal{N}$.

1. *More efficient construction:*



2. $\begin{bmatrix} |\overline{\pi}\rangle \\ \Pi_{\mathcal{A}^\perp}|f^{\overline{\pi}}\rangle \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{N}|\overline{\pi}\rangle \\ \Pi_{\mathcal{A}^\perp}|f^{\overline{\pi}}\rangle \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{N}|\overline{\pi}\rangle + \Pi_{\mathcal{A}^\perp}|f^{\overline{\pi}}\rangle \\ 0 \end{bmatrix} \mapsto \cdots \mapsto \begin{bmatrix} -|\overline{\pi}\rangle \\ \Pi_{\mathcal{A}^\perp}|f^{\overline{\pi}}\rangle \end{bmatrix}$
3. *Benefit:* Less weight on $\mathcal{N}$.

Input-invariant reflection transducer

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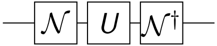
   2. $\mathcal{N}|\overline{\pi}\rangle \overset{U}{\rightsquigarrow} -\mathcal{N}|\overline{\pi}\rangle$ with complexity $R_{\mathrm{eff}}(G, r; \overline{\pi}) - 1$.
   3. Public space: $\mathcal{A} = \mathrm{Span}\{|v\rangle|*_v\rangle : v \in V\}$.
   4. Private space: $\mathcal{A}^\perp$.
2. *Problem:* depend on the input.
3. *Naive solution:*



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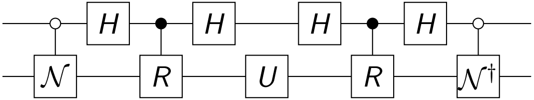

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3. *Benefit:* Less weight on $\mathcal{N}$.
4. $\Rightarrow$ smaller transduction complexity in sequential composition

Adversary bound optimization

Adversary bound optimization

1. *Result:* canonical transducer V:
   1. $|\pi\rangle \overset{V}{\rightsquigarrow} |\pi\rangle$ with complexity
      $O\left(\mathbb{E}_{v\sim\pi}\left\|C_v \wr_{|0\rangle}\right\|^2 + \mathbb{E}_{v\sim\pi}\left\|U_v \wr_{|*_v\rangle}\right\|^2\right).$
   2. $|\overline{\pi}\rangle \overset{V}{\rightsquigarrow} -|\overline{\pi}\rangle$ with complexity
      $O(R_{\text{eff}}(G, r; \overline{\pi}) - \||\overline{\pi}\rangle\|^2$
      $+ \sum_{v\in V} \frac{|\overline{\pi}_v|^2}{\pi_v}\left(\left\|C_v \wr_{|0\rangle}\right\|^2 + \left\|U_v \wr_{|*_v\rangle}\right\|^2\right)$
      $+ \sum_{v\in V}\left\|U_v \wr_{(\langle v|\otimes I)\Pi_{\mathcal{A}^\perp}|f^{\overline{\pi}}\rangle}\right\|^2\Bigg).$

Adversary bound optimization

1. *Result:* canonical transducer V:
   1. $|\pi\rangle \overset{V}{\rightsquigarrow} |\pi\rangle$ with complexity
   $$O\left(\mathop{\mathbb{E}}_{v\sim\pi}\left\|C_v \wr_{|0\rangle}\right\|^2 + \mathop{\mathbb{E}}_{v\sim\pi}\left\|U_v \wr_{|*_v\rangle}\right\|^2\right).$$
   2. $|\overline{\pi}\rangle \overset{V}{\rightsquigarrow} -|\overline{\pi}\rangle$ with complexity
   $$O(R_{\mathrm{eff}}(G, r; \overline{\pi}) - \||\overline{\pi}\rangle\|^2$$
   $$+ \sum_{v\in V} \frac{|\overline{\pi}_v|^2}{\pi_v}\left(\left\|C_v \wr_{|0\rangle}\right\|^2 + \left\|U_v \wr_{|*_v\rangle}\right\|^2\right)$$
   $$+ \sum_{v\in V}\left\|U_v \wr_{(\langle v|\otimes I)\Pi_{\mathcal{A}^\perp}|f^{\overline{\pi}}\rangle}\right\|^2\right).$$
2. *Optimization:*
   1. *Convert:* transducer $\rightarrow$ adversary sln

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   2. $|\overline{\pi}\rangle \overset{V}{\rightsquigarrow} -|\overline{\pi}\rangle$ with complexity
      $$O(R_{\mathrm{eff}}(G,r;\overline{\pi}) - \||\overline{\pi}\rangle\|^2 + \sum_{v\in V}\frac{|\overline{\pi}_v|^2}{\pi_v}\left(\left\|C_v \wr_{|0\rangle}\right\|^2 + \left\|U_v \wr_{|*_v\rangle}\right\|^2\right) + \sum_{v\in V}\left\|U_v \wr_{(\langle v|\otimes I)\Pi_{\mathcal{A}^\perp}|f^{\overline{\pi}}\rangle}\right\|^2\right).$$
2. *Optimization:*
   1. *Convert:* transducer $\to$ adversary sln

$$V \wr_{|\pi\rangle} = \begin{bmatrix} 0 \\ \bigoplus_{v\in V}\sqrt{\pi_v}\,C_v \wr_{|0\rangle} \\ \bigoplus_{v\in V}\sqrt{\pi_v}\,U_v \wr_{|*_v\rangle} \\ \vdots \end{bmatrix}$$

$$V \wr_{|\overline{\pi}\rangle} = \begin{bmatrix} \Pi_{\mathcal{A}^\perp}\left|f^{\overline{\pi}}\right\rangle \\ \bigoplus_{v\in V}\frac{|\overline{\pi}_v|^2}{\pi_v}C_v \wr_{|0\rangle} \\ \bigoplus_{v\in V}\frac{|\overline{\pi}_v|^2}{\pi_v}U_v \wr_{|*_v\rangle} \\ \vdots \end{bmatrix}$$

Adversary bound optimization

1. *Result:* canonical transducer V:
   1. $|\pi\rangle \overset{V}{\rightsquigarrow} |\pi\rangle$ with complexity
   $$O\left(\mathop{\mathbb{E}}_{v\sim\pi}\left\|C_v \rightsquigarrow_{|0\rangle}\right\|^2 + \mathop{\mathbb{E}}_{v\sim\pi}\left\|U_v \rightsquigarrow_{|*_v\rangle}\right\|^2\right).$$
   2. $|\overline{\pi}\rangle \overset{V}{\rightsquigarrow} -|\overline{\pi}\rangle$ with complexity
   $$O(R_{\mathrm{eff}}(G, r; \overline{\pi}) - \||\overline{\pi}\rangle\|^2 + \sum_{v\in V} \frac{|\overline{\pi}_v|^2}{\pi_v}\left(\left\|C_v \rightsquigarrow_{|0\rangle}\right\|^2 + \left\|U_v \rightsquigarrow_{|*_v\rangle}\right\|^2\right) + \sum_{v\in V}\left\|U_v \rightsquigarrow_{(\langle v|\otimes I)\Pi_{\mathcal{A}^\perp}|f^{\overline{\pi}}\rangle}\right\|^2\Bigg).$$
2. *Optimization:*
   1. *Convert:* transducer $\to$ adversary sln
   2. *Optimize:* remove part of the vector

$$V \rightsquigarrow_{|\pi\rangle} = \begin{bmatrix} 0 \\ \bigoplus_{v\in V}\sqrt{\pi_v}\, C_v \rightsquigarrow_{|0\rangle} \\ \bigoplus_{v\in V}\sqrt{\pi_v}\, U_v \rightsquigarrow_{|*_v\rangle} \\ \vdots \end{bmatrix}$$

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Adversary bound optimization

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Open question: time-efficiency?

Other components: amplitude amplification & annealing

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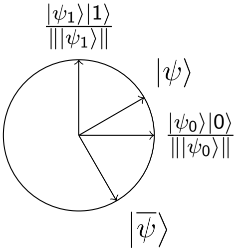
Amplitude amplification:



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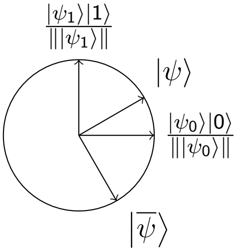

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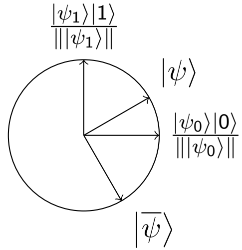

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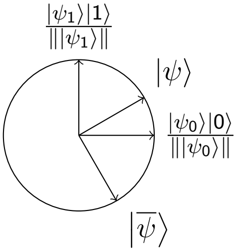

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$\Rightarrow$ *Annealing:* $|\pi\rangle \rightsquigarrow |\pi'\rangle$.

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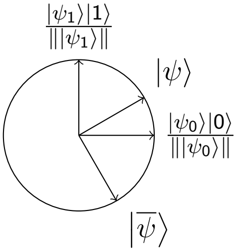

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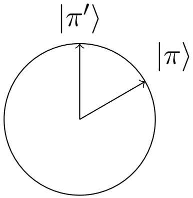
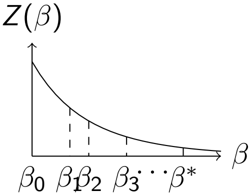


Application to volume estimation



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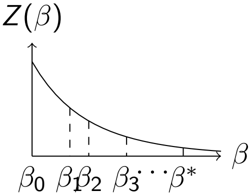

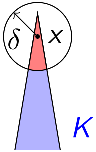

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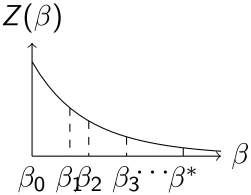

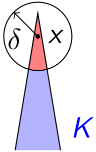

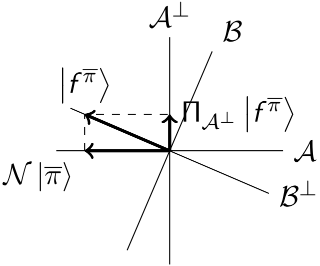

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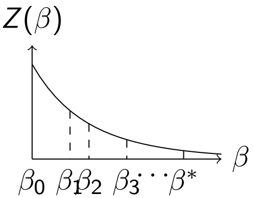

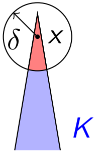

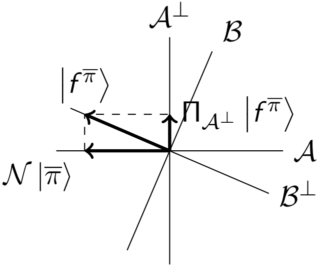

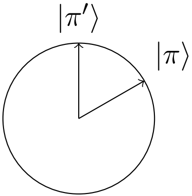

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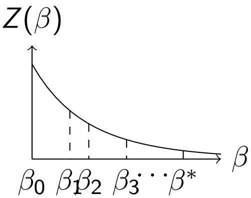

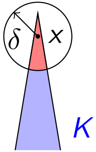

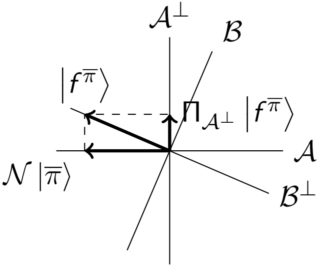

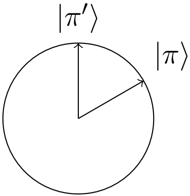

$\Rightarrow$ *$\widetilde{O}(d^3 + d^{1.75}/\varepsilon)$-query algo for vol. est.*

Thanks!

ajcornelissen@outlook.com