Quantum algorithms through composition of graphs

arXiv:2504.02115 and arXiv:2510.04973

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October 28th, 2025

Quantum algorithmic frameworks

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Setting:

1. $\mathcal{D} \subseteq \{0,1\}^n$.
2. $O_x : |j\rangle \mapsto (-1)^{x_j} |j\rangle$.
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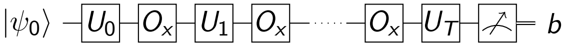
$$\mathbb{P}[b = f(x)] \geq \frac{2}{3}$$



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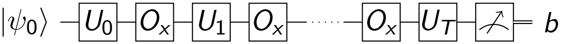

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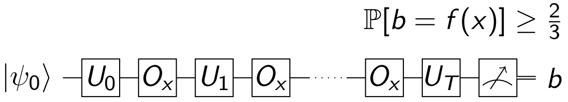

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$$f(x) = \underbrace{x_1 \wedge (x_2 \vee x_3) \vee (x_2 \wedge x_5) \vee x_3}_{\text{length } \ell}$$

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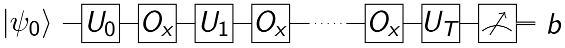

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[Rei11]

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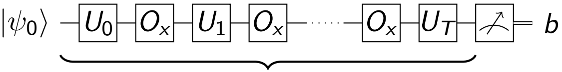

$$T = O(\sqrt{\ell})$$

Quantum algorithmic framework landscape

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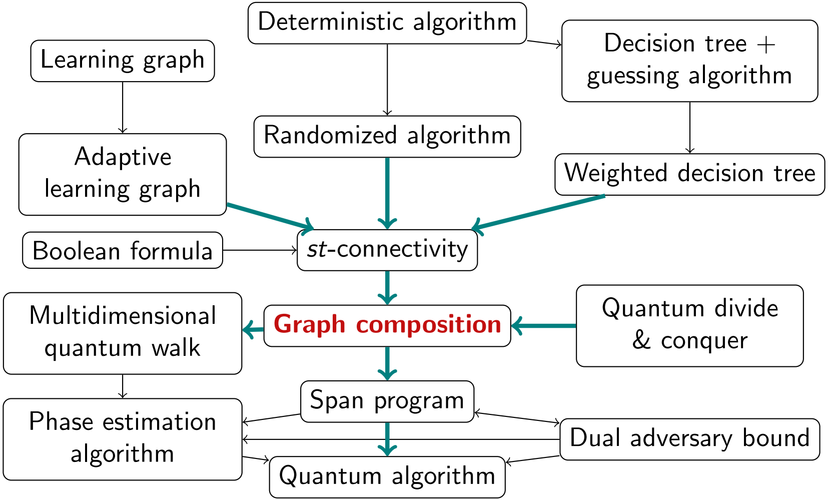
Overview:

→ Conversion between framework objects

→ New relations [This work]



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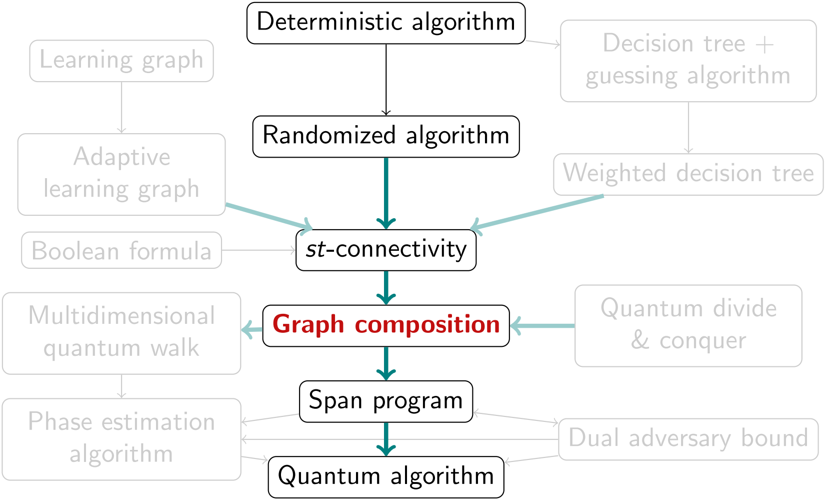
This talk:

1. Span programs
2. Graph composition
3. *st*-connectivity examples
4. Randomized → *st*-connectivity



Span programs [RŠ12,Rei09,Rei11]

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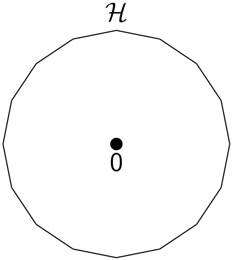

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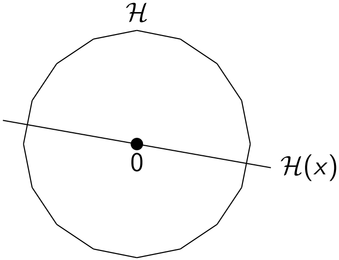

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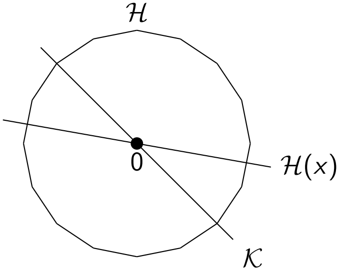

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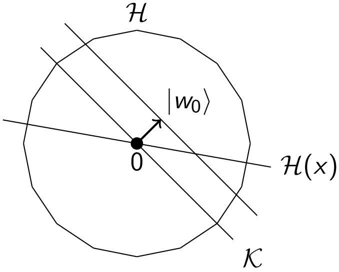

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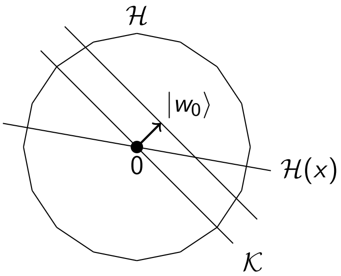

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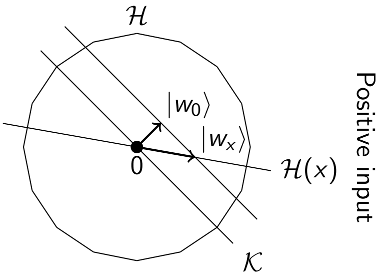

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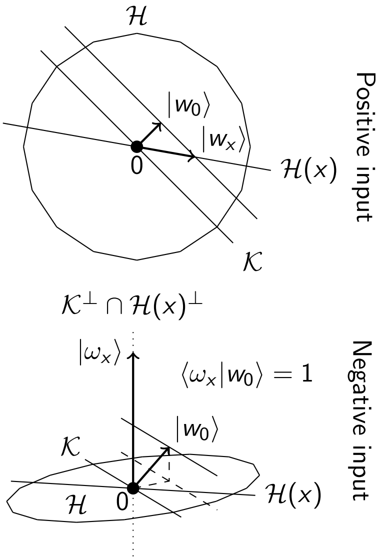

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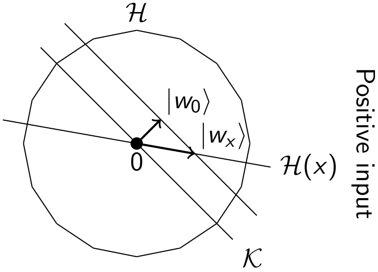
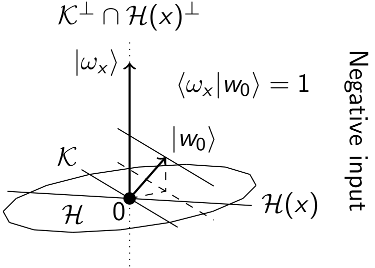
3. *Negative inputs:* $w_-(x,\mathcal{P}) = \||\omega_x\rangle\|^2$.
4. $C(\mathcal{P}) = \sqrt{\max_{x \in f^{-1}(0)} w_-(x,\mathcal{P}) \cdot \max_{x \in f^{-1}(1)} w_+(x,\mathcal{P})}$.





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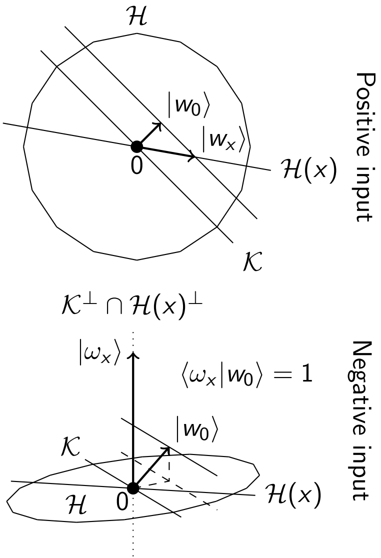
Thm: Algorithm with $O(C(\mathcal{P}))$ queries to $2\Pi_{\mathcal{H}(x)} - I$.



Span program manipulations

1. *Scalar multiplication* ($\alpha > 0$):

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2. *Trivial span program* (query x_j):
   1. $\mathcal{H} = \mathbb{C}$
   2. $\mathcal{H}(x) = \begin{cases} \mathbb{C}, & \text{if } x_j = 1, \\ \{0\}, & \text{otherwise.} \end{cases}$
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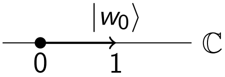

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   Then,
   1. For $x_j = 1$: $w_+(x, \mathcal{P}) = 1$.
   2. For $x_j = 0$: $w_-(x, \mathcal{P}) = 1$.

   $\Rightarrow C(x_j) = 1$.

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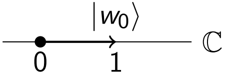

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1. Undirected graph $G = (V, E)$.
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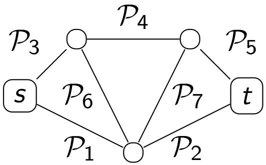

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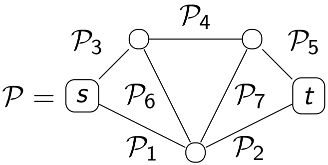

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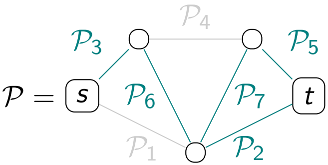

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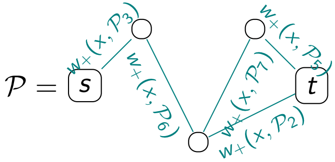

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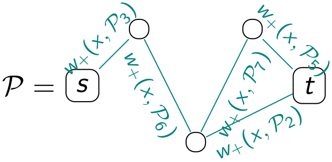

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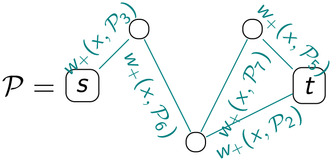

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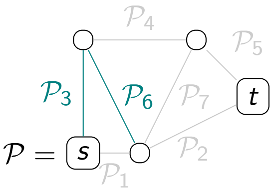

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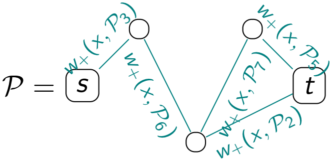

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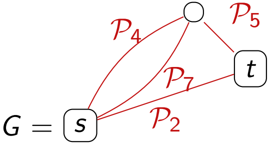

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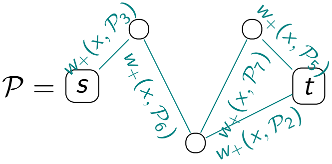
2. $w_+(x, \mathcal{P}) = R^{\mathit{eff}}_{s,t,r^+}$ with $r_+(e) = w_+(x, \mathcal{P}_e)$.

Positive input:



$$w_+(x, \mathcal{P}) = R_{G,s,t,r^+}.$$

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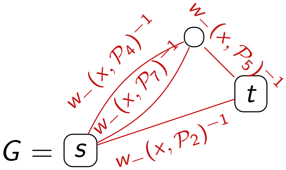

Graph composition [This work]

Ingredients:

1. Undirected graph $G = (V, E)$.
2. Source and target vertices $s, t \in V$.
3. Edge span programs $(\mathcal{P}_e)_{e \in E}$ on $\mathcal{D}$.

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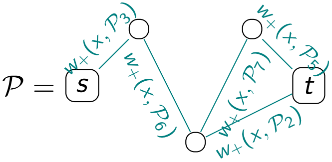

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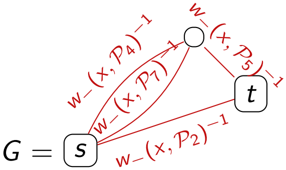

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Remark: Recovers st-connectivity with just trivial span programs. [BR12,JK17,JJKP18]

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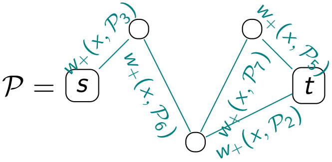

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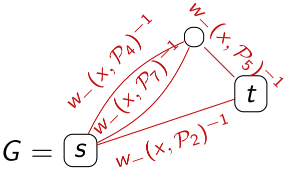

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Path-cut theorem

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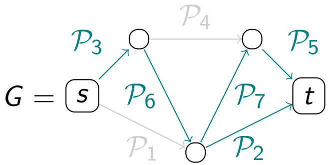

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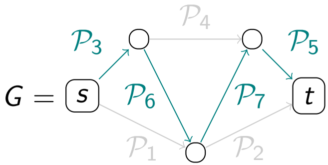

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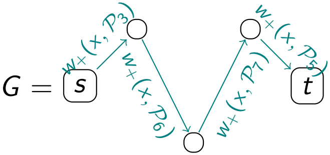

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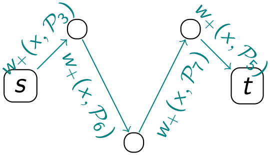
$G =$ 

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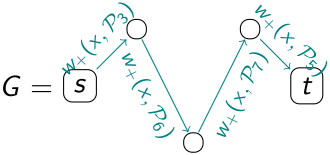

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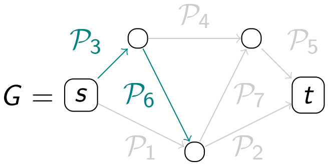

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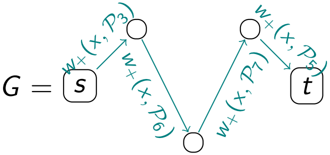

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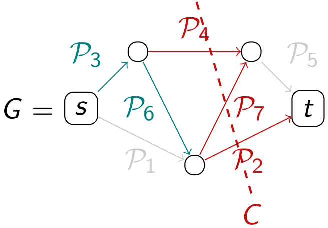

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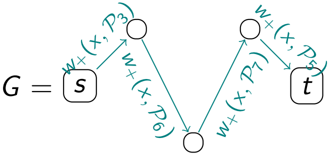

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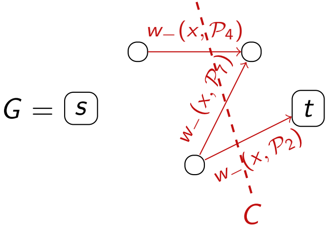

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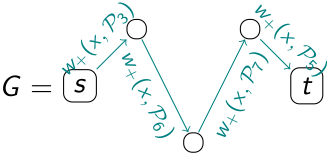

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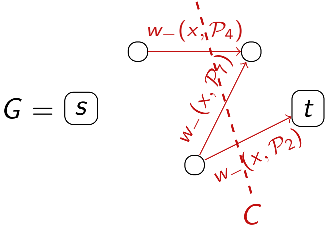

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Properties:

1. Simpler (less-powerful) version.
2. Still powerful enough for many applications.

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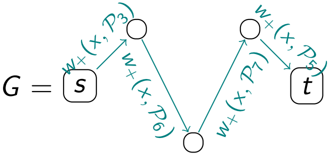

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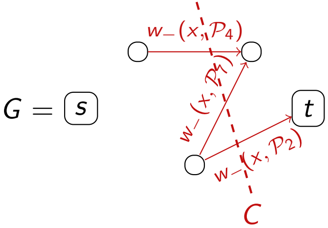

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Example: OR-function

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The OR-function:

1. $\mathrm{OR}_n : \{0,1\}^n \to \{0,1\}$

   $$\mathrm{OR}_n(x) = \begin{cases} 1, & \text{if } |x| \geq 1, \\ 0, & \text{if } |x| = 0. \end{cases}$$

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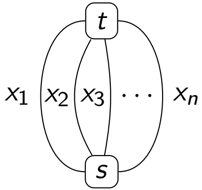

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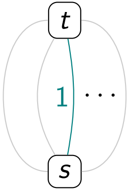

$x = 0010\cdots0 \Rightarrow w_+(x) \leq 1$

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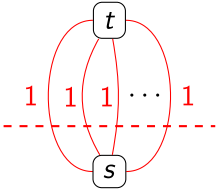

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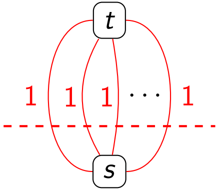

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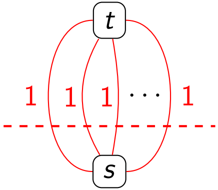

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Quadratic speed-up for search.

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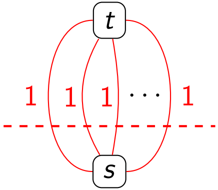

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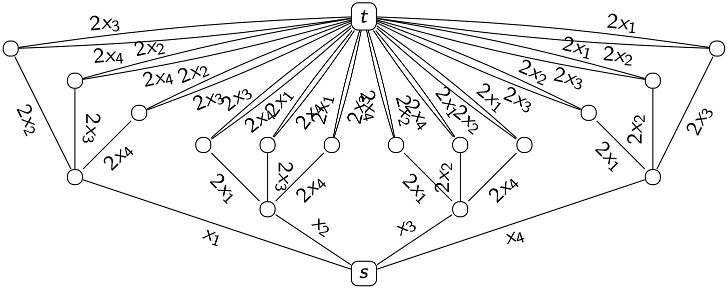
Graph composition for Th_4^3*:*



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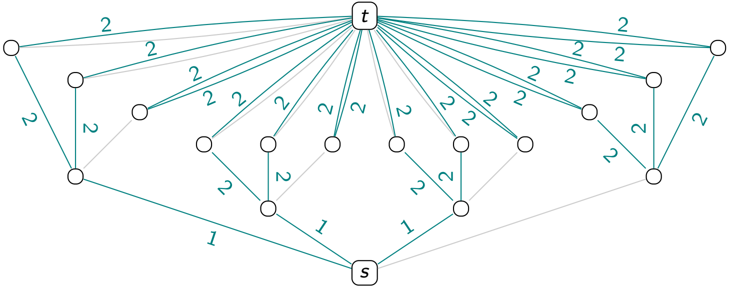

$x = 1110 \Rightarrow w_+(x) = 1$

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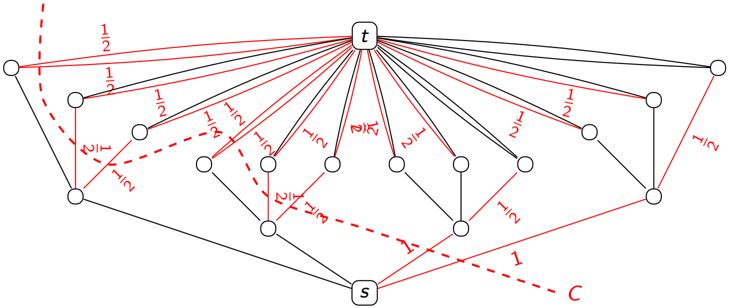

$x = 1100 \Rightarrow w_-(x) = 6$

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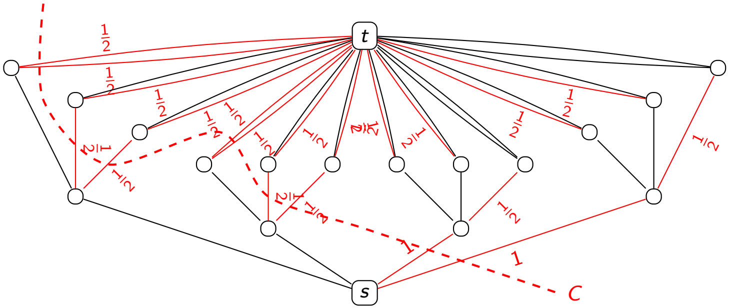

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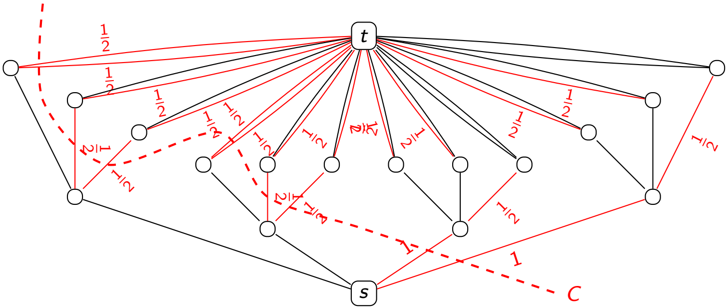

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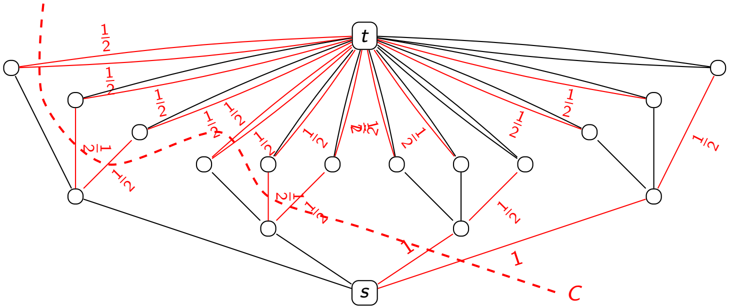

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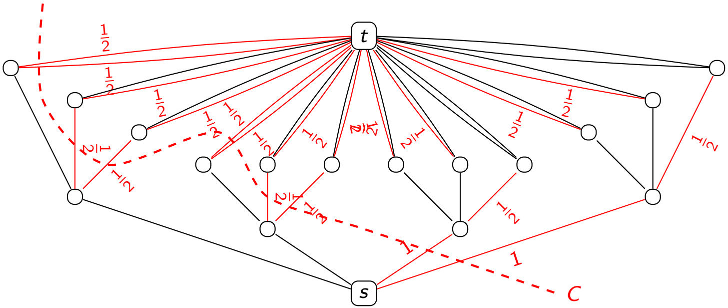
Known to be optimal!

Remark: With $k = n/2$, it also solves gapped majority in $O(1)$ queries.

Graph composition for Th_4^3*:*



$x = 1100 \Rightarrow w_-(x) = 6$

Classical algorithms → *st*-connectivity

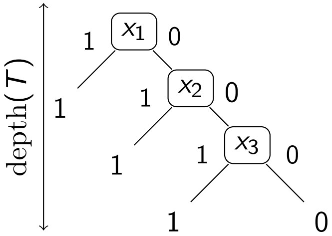
Classical algorithms $\to$ *st*-connectivity

Deterministic $\to$ st-connectivity:

Classical algorithms → *st*-connectivity

Deterministic → st-connectivity:

1. Decision tree T.



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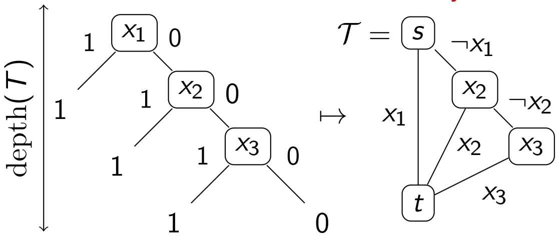
Decision tree $\to$ st-connectivity:



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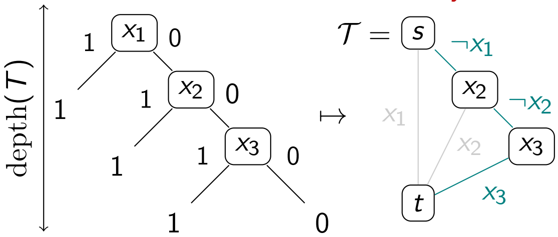

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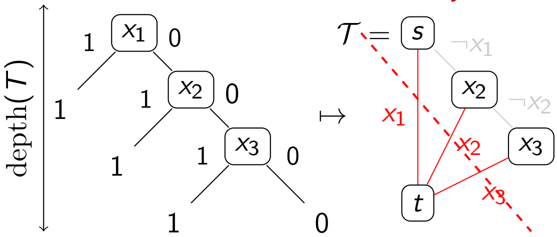

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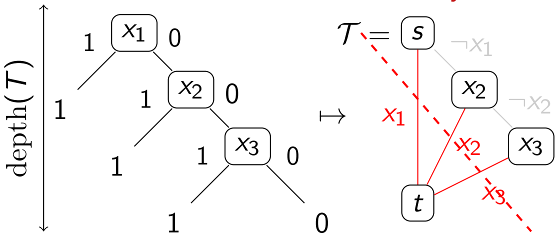

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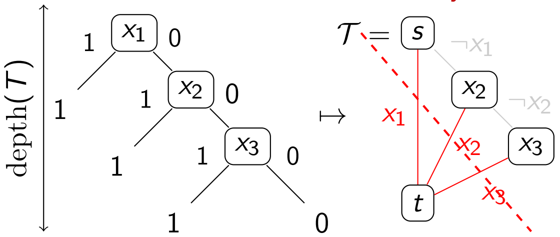

Randomized $\to$ st-connectivity: (sketch)

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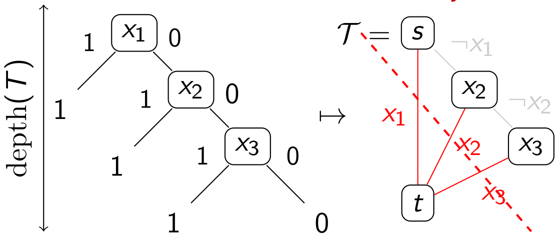

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2. *Wlog:* uniform distribution.
   $\Rightarrow$ gapped majority on N bits.

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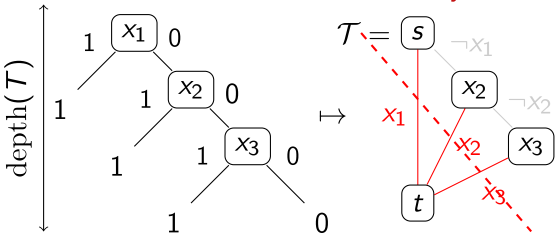

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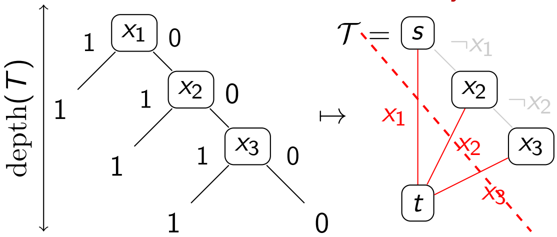

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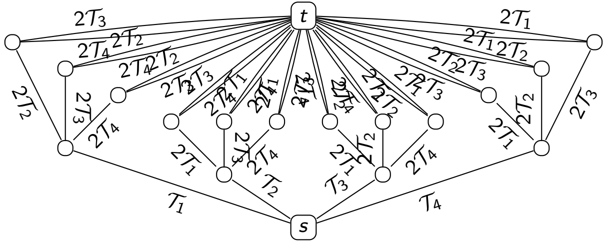

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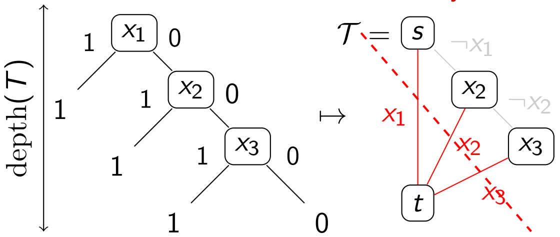
3. $C(\mathcal{P}) \in O(1 \cdot \max_j \mathrm{depth}(T_j))$.

Decision tree → st-connectivity:



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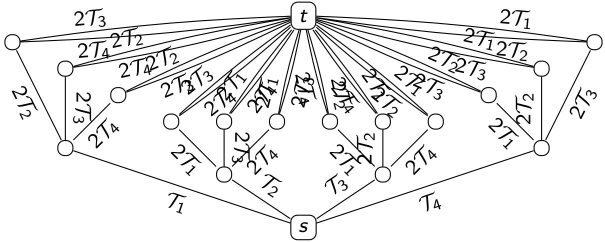

Summary

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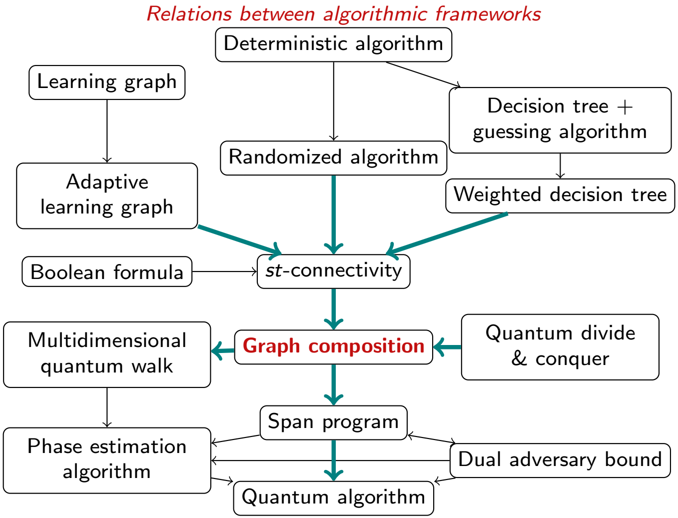
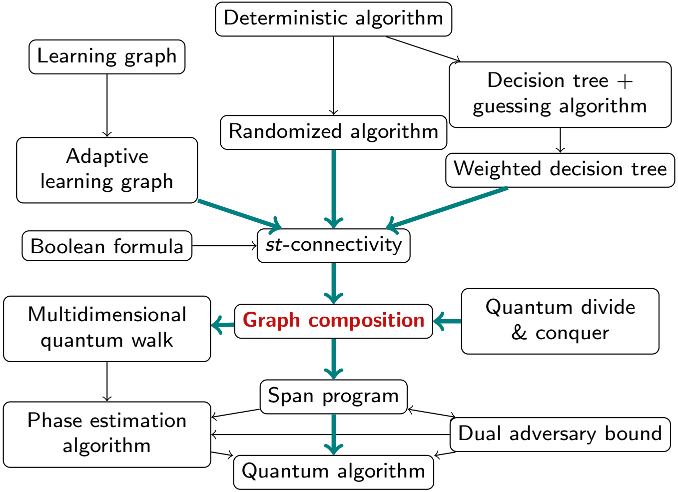

Relations between algorithmic frameworks
Deterministic algorithm
Learning graph
Decision tree + guessing algorithm
Randomized algorithm
Adaptive learning graph
Weighted decision tree
Boolean formula
st-connectivity
Multidimensional quantum walk
Graph composition
Quantum divide & conquer
Span program
Phase estimation algorithm
Dual adversary bound
Quantum algorithm

Summary

Relations between algorithmic frameworks

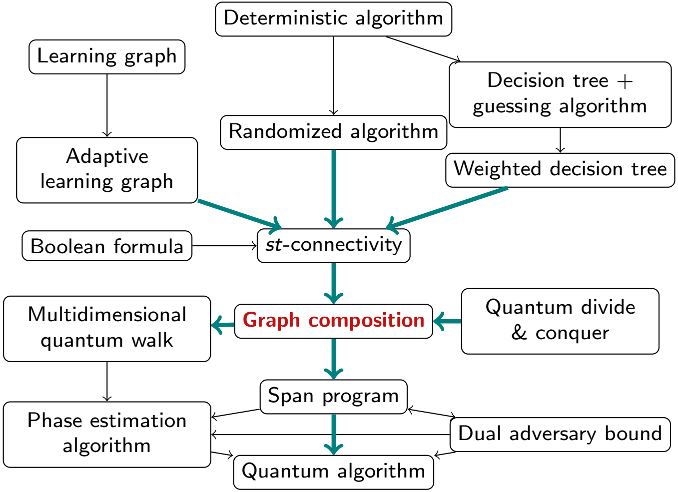


In this talk:

1. Span programs
2. Graph composition
3. Examples
4. Randomized $\to$ *st*-connectivity.

Summary

Relations between algorithmic frameworks



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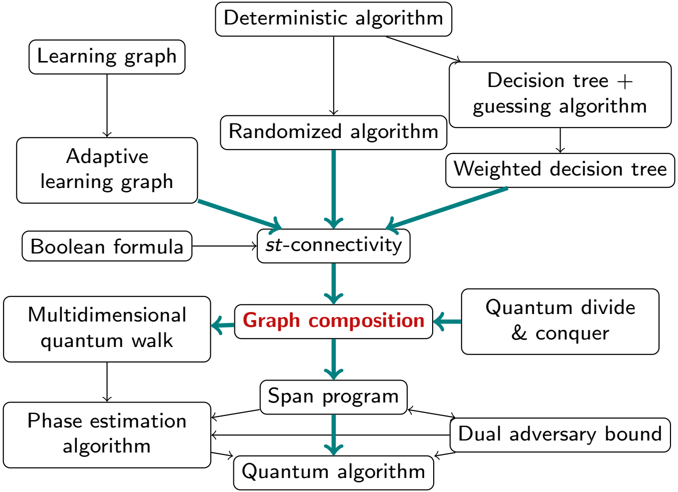

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Open question:

Limitations of *st*-connectivity?

Thanks!

Thanks for your attention!

`ajcornelissen@outlook.com`